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GENERIC SINGULARITIES OF SYMPLECTIC AND QUASI-SYMPLECTIC IMMERSIONS

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ABSTRACT. For any $k < 2n$ we construct a complete system of invariants in the problem of classifying singularities of immersed k-dimensional submanifolds of a symplectic $2n$-manifold at a generic double point.

1. INTRODUCTION

The local classification of k-dimensional immersed submanifolds of a symplectic manifold (M^{2n}, ω) is the same problem as the classification of tuples

$$\left(M^{2n}, \omega, N\right)_p, \quad N = S_1^k \cup \cdots \cup S_r^k, \tag{1.1}$$

where S_i^k are k-dimensional submanifolds of M^{2n} (strata), $p \in N$ and the notation $(\ \ )_p$ means that all objects in the parenthesis are germs at p.

Definition 1.1. A tuple $\left(M^{2n}, \omega, N\right)_p$ is equivalent, or diffeomorphic, to a tuple $\left(\tilde{M}^{2n}, \tilde{\omega}, \tilde{N}\right)_{\tilde{p}}$ if there exists a local diffeomorphism $\Phi : (M^{2n}, p) \to (\tilde{M}^{2n}, \tilde{p})$ which brings $\tilde{\omega}$ to ω and N to $\tilde{N}$.

All objects are assumed to be smooth or real-analytic. In what follows we will assume that the immersed submanifold N is *symplectic* if k is even or *quasi-symplectic* if k is odd, i.e. the following condition holds:

(G1) the restriction of ω to the tangent bundle to each of the strata S_i^k has the maximal possible rank $2[k/2]$.

The Darboux-Givental theorem (see [AG]) states that in the problem of classifying germs at $0 \in \mathbb{R}^{2n}$ of pairs consisting of a symplectic

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form on $\mathbb{R}^{2n}$ and a smooth submanifold of $\mathbb{R}^{2n}$ the pullback of the symplectic form to the submanifold is a complete invariant. This theorem implies that *if* $r = 1$ *then under the assumption* (G1) *all tuples (1.1) are equivalent.*

The present work is devoted to double points of immersed submanifolds of a symplectic manifold, i.e. we work with the tuples

$$\left(M^{2n},\ \omega,\ S_1^k \cup S_2^k\right)_p. \tag{1.2}$$

The cases $k = 1$ and $k = 2n - 1$ are much simpler than the case $2 \leq k \leq 2n - 2$. In these cases assumption (G1) always holds. They are the only cases such that all generic germs are equivalent. Here "generic" requires the following assumptions:

(**G2**) The couple $(S_1^k, S_2^k)_p$ is regular. This means that $T_pS_1^k \cap T_pS_2^k = \{0\}$ if $k \leq n$ and $T_pS_1^k + T_pS_2^k = T_pM^{2n}$ if $k > n$.

Condition (G2) implies that in the case $k \leq n$ one has $\dim T_pS_1^k + \dim T_pS_2^k = 2k$ and in the case $k > n$ one has $\dim(T_pS_1^k \cap T_pS_2^k) = 2(k - n)$.

(G3) If $k \leq n$ then the restriction of ω to the space $T_pS_1^k + T_pS_2^k$ has maximal possible rank $2k$. If $k > n$ then the restriction of ω to the space $T_pS_1^k \cap T_pS_2^k$ has maximal possible rank $2(k - n)$.

Theorem A1. *All germs of immersed* 1*-dimensional submanifolds of a symplectic* $2n$*-manifold at a double point satisfying* (G2) *and* (G3) *are equivalent.*

Remark 1.2. Theorem A1 is a particular (and the simplest) case of the symplectic classification of curves diffeomorphic to $A_k = \{x \in \mathbb{R}^{2n} : x_1^{k+1} - x_2^2 = x_{\geq 3} = 0\}$ obtained in [A], see also [DJZ2].

Theorem A2. *All germs of immersed* $(2n-1)$*-dimensional submanifolds of a symplectic* $2n$*-manifold at a double point satisfying* (G2) *and* (G3) *are equivalent.*

For any other dimensions $(k, 2n)$ the classification problem involves real or functional invariants which are constructed in sections 2 and 3.

In section 2 we associate to a generic tuple (1.2) a tuple of s complex numbers, closed with respect to the complex conjugacy, where

$$s = \min\left([k/2],\ [(2n-k)/2]\right). \tag{1.3}$$

We call them *characteristic numbers.* Theorem B states that if $2 \leq k \leq n$ then under certain genericity assumptions (including (G1) - (G3)) the tuple of characteristic numbers is a complete invariant, i.e. two tuples (1.2) are equivalent if and only if their characteristic numbers coincide.

In section 3 we extend Theorem B to the case $n < k \leq 2n-2$. In this case under the assumptions (G2) and (G3) the intersection of the strata $Q = S_1^k \cap S_2^k$ is a smooth manifold of dimension $2(k-n)$ endowed with the symplectic form $\omega_Q = \omega|_{TQ}$. By Theorem C, under certain genericity assumptions (including (G1) - (G3)) a complete invariant is a tuple of $s = [(2n-k)/2]$ germs of Hamiltonians on Q defined up to the same local symplectomorphism of (Q, ω_Q). We call these Hamiltonians *the characteristic Hamiltonians.*

In the problem of classifying generic tuples of s germs of Hamiltonians there are functional moduli if $s \geq 2$. If $s = 1$ (i.e. $n < k = 2n-2$ or $n < k = 2n-3$) then there is only one real modulus, the value of the Hamiltonian at the source point of the germ. Therefore Theorems B and C imply:

Theorem 1.3. *Let $2 \leq k \leq 2n-2$. The number $m(k, 2n)$ of moduli in the classification of generic germs of immersed k-dimensional submanifolds of a $2n$-dimensional symplectic manifold at a double point is as follows:*
$m(k, 2n) = [k/2]$ *if* $2 \leq k \leq n$; $\quad m(2n-3, 2n) = m(2n-2, 2n) = 1$;
$m(k, 2n) = \infty$ *if* $n < k \leq 2n-4$;

Note that the case $n < k \leq 2n-4$, the case of functional moduli, is possible only if $2n \geq 10$.

In section 4 we prove the algebraic statements used in the construction of the characteristic numbers. In the same section we prove the algebraic part of Theorem B. The proof uses certain result from [GZ] on the classification of couples of symplectic forms on the same vector space.

The normal forms following from Theorems A1,A2,B,C are given in section 5.

The proof of Theorem B is completed in section 6 by the linearization theorem reducing the classification of tuples (1.2) to the classification of their linearizations. The linearization theorem is proved by the method of algebraic restrictions developed in [Z] and [DJZ2]. In section 6 we also prove Theorem A1.

The proofs of Theorems C and A2 are given in sections 7 and 8. Conceptually the proofs are the same as those of Theorems B and A1, but technically they are substantially more difficult since the reduction steps in the proofs involve the linearization and the reduced linearization along the intersection of the strata $S_1^k \cap S_2^k$ which is a single point if $k \leq n$ and a $2(k-n)$dimensional manifold if $k > n$.

In the Appendix we show that our results can be extended to pairs of submanifolds of a symplectic manifold of different dimensions, i.e. tuples $(M^{2n}, \omega, S_1^{k_1} \cup S_2^{k_2})_p$ where $k_1 \neq k_2$.

2. Characteristic numbers. Theorem B

In this section we construct invariants of tuples (1.2) which we call the characteristic numbers. We present certain genericity assumptions under which in the case $2 \leq k \leq n$ the tuple of characteristic numbers is a *complete* invariant.

Definition 2.1. The linearization of the tuple (1.2) is the tuple $(T_pM^{2n}, \omega(p), T_pS_1^k \cup T_pS_2^k)$.

Introduce the following (linear) equivalence of tuples (V, μ, U) consisting of a vector space V, a 2-form μ on this space, and the union U of some subspaces of V.

Definition 2.2. A tuple (V, μ, U) is equivalent to a tuple $(\tilde{V}, \tilde{\mu}, \tilde{U})$ if there exists a liner bijection $L : V \to \tilde{V}$ such that $L^*\tilde{\mu} = \mu$ and $L(U) = \tilde{U}$.

Proposition 2.3. *If two tuples* $(M^{2n}, \omega, S_1^k \cup S_2^k)_p$ *and* $(\tilde{M}^{2n}, \tilde{\omega}, \tilde{S}_1^k \cup \tilde{S}_2^k)_{\tilde{p}}$ *are equivalent then their linearizations are equivalent.*

Proof. If the two tuples are equivalent via a local diffeomorphism Φ then their linearizations are equivalent via the linear transformation $L = d\Phi(p)$. □

Now we construct the *reduced linearization.* If k is odd, introduce the lines

$$k \text{ odd}: \quad \ell_1 = \ker \omega|_{T_pS_1^k}, \ \ \ell_2 = \ker \omega|_{T_pS_2^k}$$

(they are lines under the assumption (G1)) and introduce the vector space

$$k \text{ even}: \qquad W = \begin{cases} T_pS_1^k + T_pS_2^k & \text{if } k \leq n \\ \left(T_pS_1^k \cap T_pS_2^k\right)^\omega & \text{if } k > n; \end{cases}$$

$$k \text{ odd}: \qquad W = \begin{cases} \left(T_pS_1^k + T_pS_2^k\right) \cap (\ell_1 + \ell_2)^\omega & \text{if } k \leq n \\ \left(T_pS_1^k \cap T_pS_2^k\right)^\omega \cap (\ell_1 + \ell_2)^\omega & \text{if } k > n. \end{cases}$$

Here the sign $^\omega$ denotes the skew-orthogonal complement in the symplectic vector space $(T_pM^{2n}, \omega(p))$. Set, for any parity of k

$$\sigma = \omega|_W, \quad U_1 = T_pS_1^k \cap W, \quad U_2 = T_pS_2^k \cap W.$$

Definition 2.4. The tuple $(W, \sigma, U_1 \cup U_2)$ will be called the reduced linearization of the tuple (1.2).

Proposition 2.5. *The equivalence of the linearizations of two tuples (1.2) implies the equivalence of their reduced linearizations.*

Proof. If the two linearizations are equivalent via a linear transformation L then the reduced linearizations are equivalent via the restriction of L to the space W. □

Note that the reduced linearization coincides with the linearization in the only case that $k = n$ is an even number. For all other dimensions the reduced linearization is simpler than linearization provided certain genericity assumptions. We will assume (G1) - (G3) from section 1 and two more conditions:

(**G4**) if k is odd then ω does not annihilate the 2-plane $\ell_1 + \ell_2$.

(**G5**) the space $(T_pS_1^k)^\omega$ is transversal to $T_pS_2^k$ in T_pM^{2n}.

Remark 2.6. The fact that $\ell_1 \neq \ell_2$ follows from (G2) if $k \leq n$ and from (G3) if $k > n$. If $k = 1$ or $k = 2n - 1$ then (G4) and (G5) follow from (G1) - (G3).

Proposition 2.7. *Under the assumptions* (G1) - (G5) *the reduced linearization has the following properties where $s = s(k, 2n)$ is the integer defined by (1.3):*

(a) *W^{4s} is a $4s$-dimensional vector space;*

(b) *σ is a symplectic form on W;*

(c) *U_1 and U_2 are transversal symplectic $2s$-dimensional subspaces of (W, σ).*

(d) *The space U_1^σ is transversal to U_2 in W.*

Here the sign $^\sigma$ denotes the skew-orthogonal complement in the symplectic space (W, σ). Proposition 2.7 is proved in section 4.

Remark 2.8. If $k = 1$ or $k = 2n - 1$ then $s(k, 2n) = 0$ and under assumptions (G1) - (G3) (implying (G4) and (G5)) the reduced linearization of (1.2) is the "zero tuple": $W = U_1 = U_2 = \{0\}$, $\sigma = 0$.

The next step is the construction of two linear operators associated with the reduced linearization $(W, \sigma, U_1 \cup U_2)$ satisfying (a) - (d) in Proposition 2.7. Consider the following direct sums and the corresponding projections:

$$W = U_1 \oplus U_1^\sigma, \quad \pi_1 : W \to U_1,$$

$$W = U_2 \oplus U_2^\sigma, \quad \pi_2 : W \to U_2.$$

Define linear operators $T_1 : U_1 \to U_1$ and $T_2 : U_2 \to U_2$ by the diagram

$$\begin{array}{ccccccc} U_1 & & \xrightarrow{T_1} & & U_1 & & \\ & \searrow_{\pi_2} & & {}^{\pi_1}\nearrow & & \searrow_{\pi_2} & \\ & & U_2 & \xrightarrow[T_2]{} & & & U_2 \end{array} : \qquad \begin{array}{c} T_1 = \pi_1 \circ (\pi_2|_{U_1}) \\ \\ T_2 = \pi_2 \circ (\pi_1|_{U_2}) \end{array}$$

Lemma 2.9. *Under conditions* (G1)- (G5) *the linear operators* T_1 *and* T_2 *are conjugate and consequently have the same eigenvalues.*

Proof. The diagram above implies that the diagram

$$\begin{array}{ccc} U_1 & \xrightarrow{T_1} & U_1 \\ \pi_2 \downarrow & & \downarrow \pi_2 \\ U_2 & \xrightarrow[T_2]{} & U_2 \end{array}$$

is commutative. Items (c) and (d) in Proposition 2.7 imply that the three spaces U_1, U_2, U_1^σ are transversal one to the other. It follows that π_2 restricted to U_1 is a bijection between U_1 and U_2. □

Definition 2.10. The eigenvalues of the operator T_1 will be called the characteristic numbers (of the tuple (1.2) or of its linearization or of its reduced linearization).

Proposition 2.11. *If two reduced linearizations* $(W, \sigma, U_1 \cup U_2)$ *and* $(\tilde{W}, \tilde{\sigma}, \tilde{U}_1 \cup \tilde{U}_2)$ *satisfy (a) - (d) in Proposition 2.7 and are equivalent then they have the same characteristic numbers.*

Proof. Let T_1, T_2 and $\tilde{T}_1, \tilde{T}_2$ be the linear operators associated with the reduced linearizations. Since their construction is canonical, the equivalence of the reduced linearizations implies that T_1 is conjugate with one of the operators $\tilde{T}_1, \tilde{T}_2$. Now the proposition follows from Lemma 2.9. □

The following statement is a logical corollary of Propositions 2.3, 2.5, and 2.11:

Proposition 2.12. *If two tuples* $(M^{2n}, \omega, S_1^k \cup S_2^k)_p$ *and* $(\tilde{M}^{2n}, \tilde{\omega}, \tilde{S}_1^k \cup \tilde{S}_2^k)_{\tilde{p}}$ *satisfy* (G1) - (G5) *and are equivalent then they have the same characteristic numbers.*

Since the operator T_1 is defined on the $2s$-space, $s = s(k, 2n)$, one may think that for a generic tuple (1.2) there are $2s$ characteristic numbers. This is not so.

Proposition 2.13. *Consider a tuple* $(W, \sigma, U_1 \cup U_2)$ *satisfying* (a) - (d) *in Proposition 2.7. Each of the eigenvalues of the associated operator* T_1 *has multiplicity* ≥ 2. *If the tuple is generic then each of the eigenvalues has multiplicity* 2. *Consequently there are not more than* $s = s(k, 2n)$ *characteristic numbers where* $s(k, 2n)$ *is defined by (1.3), and for a generic tuple* $(W, \sigma, U_1 \cup U_2)$ *there are exactly* $s(k, 2n)$ *characteristic numbers.*

The proof of this proposition is contained in section 4 and its explanation is as follows: the matrix of the operator T_1 in some (and then any) basis of the space U_1 is the product of two skew-symmetric matrices.

In view of Proposition 2.13 we introduce the last genericity assumption:

(**G6**) If $4 \leq k \leq 2n - 4$ so that $s(k, 2n) \geq 2$ then the number of characteristic numbers is maximal possible, i.e. $s(k, 2n)$.

Theorem B. *Let* $2 \leq k \leq n$. *In the problem of classifying germs of immersed* k*-dimensional submanifolds of a symplectic* $2n$*-manifold at a double point satisfying* (G1)-(G6) *the tuple of characteristic numbers is a complete invariant.*

Remark 2.14. In the case $k = 2$ Theorem B is covered by our classification in [DJZ2] section 7.4, requiring only the assumptions (G1)-(G3). This classification involves an invariant which we called the index of non-orthogonality between the strata S_1^2 and S_2^2. Under assumption (G5) the index of non-orthogonality and the characteristic number are the same invariant.

3. Characteristic Hamiltonians. Theorem C

To extend theorem B to the case $n < k \leq 2n - 2$ consider (for such dimensions and under assumptions (G1)-(G3)) the symplectic manifold

$$(Q, \omega_Q) = \big(S_1^k \cap S_2^k, \ \ \omega|_{T(S_1^k \cap S_2^k)}\big) \tag{3.1}$$

and consider, along with the tuple (1.2) the family of tuples

$$(M^{2n}, \omega, S_1^k \cup S_2^k)_q, \quad q \in Q \tag{3.2}$$

which are the germs of the same tuple (1.2), but at points $q \in Q$, close to p. It is clear that if (1.2) satisfies (G1)-(G6) then so does (3.2), for any point $q \in Q$ close to p. Therefore under (G1)-(G6) we have for any point $q \in Q$ a tuple

$$\lambda_{q,1}, \ldots \lambda_{q,s}, \quad s = s(k, 2n)$$

of characteristic numbers of the tuple (3.2). We obtain $s = s(k, 2n)$ function germs:

$$H_i : (Q, p) \to (\mathbb{R}, \lambda_i), \quad H_i(q) = \lambda_{q,i}, \qquad i = 1, ..., s = s(k, 2n),$$

where λ_i are the characteristic numbers of (1.2).

Definition 3.1. The constructed function germs H_i will be called the characteristic Hamiltonians associated with the tuple (1.2).

Note that the characteristic Hamiltonians are constructed only for the case $n < k \leq 2n-2$ and under assumptions (G1)- (G6). Consider now two tuples

$$(M^{2n}, \omega, S_1^k \cup S_2^k)_p, \quad (\tilde{M}^{2n}, \tilde{\omega}, \tilde{S}_1^k \cup \tilde{S}_2^k)_{\tilde{p}} \tag{3.3}$$

satisfying (G1)-(G6). Let $H_i : (Q, p) \to (\mathbb{R}, \lambda_i)$ and $\tilde{H}_i : (\tilde{Q}, \tilde{p}) \to (\mathbb{R}, \tilde{\lambda}_i)$ be the characteristic Hamiltonians associated with (3.3).

Proposition 3.2. *Let $n \leq k \leq 2n-2$. If the tuples (3.3) satisfy* (G1)-(G6) *and are equivalent then there exists a local diffeomorphism $\phi : (Q, p) \to (\tilde{Q}, \tilde{p})$ which brings $\tilde{\omega}_{\tilde{Q}}$ to ω_Q and such that $(\tilde{H}_1, ..., \tilde{H}_s) \circ \phi = (H_1, ..., H_s)$.*

Proof. Assume that the tuples (3.3) are equivalent via a local diffeomorphism Φ. Let ϕ be the restriction of Φ to Q. It is clear that ϕ sends Q to $\tilde{Q}$ and $\tilde{\omega}_{\tilde{Q}}$ to ω_Q. The tuple $(M^{2n}, \omega, S_1^k \cup S_2^k)_q$ is equivalent to the tuple $(\tilde{M}^{2n}, \tilde{\omega}, \tilde{S}_1^k \cup \tilde{S}_2^k)_{\phi(q)}$. Now Proposition 3.2 follows from Proposition 2.12. □

Proposition 3.2 means that in the problem of classifying tuples (1.2) satisfying (G1)-(G6), the tuple of characteristic Hamiltonians defined up to a symplectomorphism of (Q, ω_Q) is an invariant. We claim that this invariant is complete.

Theorem C. *Let $n < k \leq 2n-2$. In the problem of classifying germs of immersed k-dimensional submanifolds of a symplectic $2n$-manifold at a double point satisfying* (G1)-(G6)*, a complete invariant is the tuple of characteristic Hamiltonians on the symplectic manifold (3.1) defined up to a symplectomorphism of this manifold.*

If $k = 2n-2$ or $k = 2n-3$ then $s(k, 2n) = 1$ and there is only one characteristic Hamiltonian $H = H_1 : (Q, p) \to (\mathbb{R}, \lambda_1)$ associated with the tuple (1.2). Here λ_1 is the only characteristic number. In this case the assumption (G6) always holds. Introduce the genericity assumption

(**G7**) If $n < k = 2n-2$ or $n < k = 2n-3$ then the characteristic Hamiltonian $H : (Q, p) \to (\mathbb{R}, \lambda_1)$ is non-singular, i.e. $dH(p) \neq 0$.

It is well known that if f and g are non-singular function germs at the same point p of a symplectic manifold and $f(p) = g(p)$ then f can be brought to g by a local symplectomorphism. Therefore Theorem **C** implies:

Theorem 3.3 (Corollary of Theorem C). *Let $n < k = 2n-2$ or $n < k = 2n-3$. In the problem of classifying germs of immersed k-dimensional submanifolds of a symplectic $2n$-manifold at a double point satisfying the assumptions* (G1)-(G5) *and* (G7) *the characteristic number $\lambda = \lambda_1$ is a complete invariant.*

If $n < k < 2n-3$ then $s = s(k, 2n) \geq 2$ and Theorem C implies that in the classification of singularities of symplectic or quasi-symplectic immersions there are functional moduli: the functional moduli in the classification of s-tuples of Hamiltonians on a fixed symplectic space.

4. Algebraic part

In this section we prove Propositions 2.7 and 2.13 and also we prove:

Proposition 4.1. *If the reduced linearizations of two tuples (1.2) satisfying the assumptions* (G1) - (G4) *are equivalent then their linearizations are equivalent.*

Proposition 4.2. *If the reduced linearizations of two tuples (1.2) satisfying the assumptions* (G1) - (G6) *have the same characteristic numbers then these reduced linearizations are equivalent.*

Note that in these statement "if" can be replaced by "only if". The "only if" part is already proved: see Propositions 2.5 and 2.11.

The proofs require the following normal form for the linearization of a tuple (1.2). Assumptions (G1) - (G3) allow us to choose local coordinates

$$x, y \in \mathbb{R}^{2s}, \ z \in \mathbb{R}^{2n-4s}, \quad s = s(k, 2n)$$

centered at the point $p \in M^{2n}$ such that the following holds:

1. the strata S_1^k, S_2^k are given by the equations in Table 1;

2. the vector spaces $T_pS_1^k + T_pS_2^k$, $T_pS_1^k \cap T_pS_2^k$, their skew-orthogonal complements and the lines ℓ_1, ℓ_2 are spanned by the vectors given in Table 1;

3. ω restricted to the space $\mathrm{span}(\partial/\partial z_i)$ has Darboux normal form $\sum dz_{2i-1} \wedge dz_{2i}$.

Table 1. Normal form for the linearization of (1.2) $s = s(k, 2n) = \min([k/2], [(2n-k)/2])$, $x, y \in \mathbb{R}^{2s}$, $z \in \mathbb{R}^{2n-4s}$.

	$k \le n$, k even	$k \le n$, k odd	$k > n$, k even	$k > n$, k odd
S_1^k	$y = z = 0$	$y = z_2 = z_{\ge 3} = 0$	$y = 0$	$y = z_2 = 0$
S_2^k	$x = z = 0$	$x = z_1 = z_{\ge 3} = 0$	$x = 0$	$x = z_1 = 0$
$T_pS_1^k + T_pS_2^k$	$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$	$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}$	T_pM^{2n}	T_pM^{2n}
$(T_pS_1^k + T_pS_2^k)^\omega$	$\frac{\partial}{\partial z_i}$	$\frac{\partial}{\partial z_{\ge 3}}$	0	0
$T_pS_1^k \cap T_pS_2^k$	0	0	$\frac{\partial}{\partial z_i}$	$\frac{\partial}{\partial z_{\ge 3}}$
$(T_pS_1^k \cap T_pS_2^k)^\omega$	T_pM^{2n}	T_pM^{2n}	$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$	$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}$
ℓ_1	0	$\frac{\partial}{\partial z_1}$	0	$\frac{\partial}{\partial z_1}$
ℓ_2	0	$\frac{\partial}{\partial z_2}$	0	$\frac{\partial}{\partial z_2}$
$(\ell_1 + \ell_2)^\omega$	T_pM^{2n}	$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_{\ge 3}}$	T_pM^{2n}	$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_{\ge 3}}$

In these coordinates one has, for each of the cases in Table 1:

$$
\begin{aligned}
\omega(p) = &-\sum_{i,j=1}^{2s} A_{ij} dx_i \wedge dx_j + \sum_{i,j=1}^{2s} B_{ij}^{-1} dy_i \wedge dy_j + \\
&+ \sum_{i,j=1}^{2s} C_{ij} dx_i \wedge dy_j + \sum_{i=1}^{n-2s} dz_{2i-1} \wedge dz_{2i},
\end{aligned}
\tag{4.1}
$$

where A, B, and C are $2s \times 2s$ matrices, the matrices A and B are skew-symmetric and non-singular. (The sign $-$ at the first sum and using B^{-1} instead of B are convenient for further calculations). The reduced linearization is the tuple $(W, \sigma, U_1 \cup U_2)$ with

$$
\begin{aligned}
W &= \mathrm{span}(\partial/\partial x_i, \partial/\partial y_i), \\
U_1 \cup U_2 &= \mathrm{span}(\partial/\partial x_i) \cup \mathrm{span}(\partial/\partial y_i), \\
\sigma &= -\sum_{i,j=1}^{2s} A_{ij} dx_i \wedge dx_j + \sum_{i,j=1}^{2s} B_{ij}^{-1} dy_i \wedge dy_j + \sum_{i,j=1}^{2s} C_{ij} dx_i \wedge dy_j.
\end{aligned}
\tag{4.2}
$$

Proof of Propositions 2.7 and 4.1. These proposition are direct corollaries of the given normal form for the linearization of tuple (1.2).

To prove Propositions 2.13 and 4.2 we need the following simple lemmas.

Lemma 4.3. *For the tuple (4.2) the condition that* $(U_1)^\sigma$ *is transversal to* U_2 *is equivalent to the condition* $\det C \neq 0$*.*

Proof. The space U_1^σ is given by the equations $Cdy - 2Adx = 0$ (here dx and dy are the columns with components dx_i and dy_i). The space U_2 is given by equations $dx = 0$, and the lemma follows. □

Lemma 4.4. *If* $\det C \neq 0$ *then the tuple (4.2) is equivalent to a tuple of the same form with* $C = I$*.*

Proof. Take a matrix Q such that $CQ = I$. The linear transformation $x \to Qx, y \to y$ brings (4.2) to a tuple of the same form with $C = I$. □

Lemma 4.5. *Assume* $C = I$*. Then the representative matrix of the linear operator* T_1 *associated with the tuple (4.2) in the basis* $(\partial/\partial x_i)$ *is the matrix* $\frac{1}{4}A^{-1}B$*.*

Proof. The spaces $(U_1)^\sigma$ and $(U_2)^\sigma$ are given by the equations

$$(U_1)^\sigma: \ dy - 2Adx = 0, \quad (U_2)^\sigma: \ dx - 2B^{-1}dy = 0.$$

It follows that in the basis $\{\partial/\partial x_i\}$ and $\{\partial/\partial y_i\}$ the matrices of the operators $\pi_1|_{U_2}$ and $\pi_2|_{U_1}$ are $-\frac{1}{2}A^{-1}$ and $-\frac{1}{2}B$ respectively, which implies the lemma. □

Proof of Propositions 2.13. By Proposition 2.11 and Lemma 4.4 we may restrict ourselves to the tuple (4.2) with $C = I$. By Lemma 4.5 the eigenvalues of the operator T_1 are the eigenvalues of the matrix $\frac{1}{4}A^{-1}B$ which is the product of two non-singular skew-symmetric $2s \times 2s$ matrices. Now Proposition 2.13 follows from the first statement of the following theorem proved in [GZ]:

Theorem 4.6 ([GZ], section 1)**.** *Let* A *and* B *be non-singular skew-symmetric* $2s \times 2s$ *matrices.*

1. The multiplicity of each of the eigenvalues of the matrix $A^{-1}B$ *is* ≥ 2 *and consequently this matrix has not more than* s *distinct eigenvalues. If* A *and* B *are generic then the matrix* $A^{-1}B$ *has exactly* s *distinct eigenvalues.*

2. The tuple of eigenvalues of the matrix $A^{-1}B$ *is an invariant of the couple* (A, B) *with respect to the group of transformations* $(A, B) \to (R^tTR,\ R^tBR)$, $\det R \neq 0$*. If the matrix* $A^{-1}B$ *has* s *distinct eigenvalues then this invariant is complete.*

Proof of Proposition 4.2. By Lemma 4.4 the tuple (4.2) is equivalent to a tuple of the same form with $C = I$. Therefore Lemma 4.5 reduces Proposition 4.2 to the following statement: if $A, B, \tilde{A}, \tilde{B}$ are non-singular skew-symmetric $2s \times 2s$ matrices and the matrices $A^{-1}B$ and $\tilde{A}^{-1}\tilde{B}$ have the same s distinct eigenvalues then the tuple (4.2) with $C = I$ is equivalent to a tuple of the same form (also with $C = I$) with A and B replaced by $\tilde{A}$ and $\tilde{B}$. Note that the linear transformation $x \to Rx, y \to (R^t)^{-1}y$ brings the tuple (4.2) with $C = I$ to the tuple of the same form with $C = I$ and A and B replaced by T^tAT and T^tBT. The existence of a non-singular matrix T such that $T^tAT = \tilde{A}$ and $T^tBT = \tilde{B}$ is exactly the second statement of Theorem 4.6.

5. Normal forms

Theorem 5.1. *A tuple $(M^{2n}, \omega, S_1^k \cup S_2^k)_p$ describing a generic germ of an immersed k-dimensional submanifold of a symplectic $(2n)$-manifold at a double point p is equivalent to the tuple $(\mathbb{R}^{2n}, \omega^*, S_1^{k,*} \cup S_2^{k,*})_0$ where $\omega^*, S_1^{k,*}, S_2^{k,*}$ are given in Table 2. The genericity assumptions are given in the first column of the table. The parameters λ_i are the characteristic numbers and consequently (by Theorem B and by Theorem 3.3) their non-ordered tuple is an invariant. The functional parameters $H_i(u, v)$ are the characteristic Hamiltonians with respect to the form $\sum du_i \wedge dv_i$ and consequently (by Theorem C) the non-ordered tuple of these Hamiltonians is an invariant up to a symplectomorphism with respect to this form.*

Proof. If $k = 1$ or $k = 2n-1$ then Theorem 5.1 follows from Theorems A1 and A2.

If $2 \leq k \leq n-2$ then the reduced linearization of the tuple $(\mathbb{R}^{2n}, \omega^*, (S_1^k)^*, (S_2^k)^*)_0$ has the form (4.2) with $C = I$ and certain skew-symmetric matrices A and B. Lemma 4.5 implies that the parameters λ_i in the normal forms are exactly the characteristic numbers.

If $n < k \leq 2n-2$ then the intersection of $(S_1^k)^*$ and $(S_2^k)^*$ is a manifold Q given by equations $x = y = 0$, and the restriction of ω^* to this manifold is the symplectic form $\sum du_i \wedge dv_i$. Let $q = (u, v) \in Q$ be a point close to $0 \in \mathbb{R}^{2n}$. The reduced linearization of the tuple $(\mathbb{R}^{2n}, \omega^*, (S_1^k)^*, (S_2^k)^*)_q$ has the form (4.2) with $C = I$ and certain skew-symmetric matrices A and B; the matrix B depends on the point $q = (u, v)$. Lemma 4.5 implies that the parameters $\lambda_i(u, v)$ in the normal forms are exactly the characteristic Hamiltonians on Q. $\square$

Remark 5.2. None of the characteristic numbers λ_i is equal to zero and consequently the characteristic Hamiltonians take non-zero values.

This fact is equivalent to the fact that the linear operator T_1 is non-singular which follows from (a)-(d) in Proposition 2.7.

Remark 5.3. The tuple of characteristic numbers is closed with respect to complex conjugacy. If $\lambda_j = \bar{\lambda}_i$ are non-real characteristic numbers then the coordinates $x_{2j-1}, x_{2j}, y_{2j-1}, y_{2j}$ and $x_{2i-1}, x_{2i}, y_{2i-1}, y_{2i}$ are complex valued and conjugate: $x_{2j-1} = \bar{x}_{2i-1}, x_{2j} = \bar{x}_{2i}, y_{2j-1} = \bar{y}_{2i-1}, y_{2j} = \bar{y}_{2i}$. The Hamiltonians $H_i(u, v)$ and $H_j(u, v)$ are also complex valued and conjugate: $H_j(u, v) = \bar{H}_i(u, v)$.

6. Linearization theorem. Proof of Theorems B and A1

Propositions 4.1 and 4.2 and the fact that under the assumption (G2) any two couples $(S_1^k, S_2^k)_p$ and $(\tilde{S}_1^k, \tilde{S}_2^k)_{\tilde{p}}$ are diffeomorphic reduce Theorem B to the following statement involving two tuples with the same strata S_1^k and S_2^k:

$$(\mathbb{R}^{2n}, \omega, S_1^k \cup S_2^k)_0, \quad (\mathbb{R}^{2n}, \tilde{\omega}, S_1^k \cup S_2^k)_0 \tag{6.1}$$

Theorem 6.1 (Linearization theorem)**.** *Let $1 \le k \le n$. If the linearizations of tuples (6.1) satisfying the assumptions (G1) and (G2) are the same, i.e. $\omega(0) = \tilde{\omega}(0)$, then these tuples are diffeomorphic.*

Theorem 6.1 also implies Theorem A1. In fact, if $k = 1$ then the assumption (G1) trivially holds, and the assumptions (G2) and (G3) in Theorem A1 imply (G4). and (G5). Therefore one can use Proposition 4.1 stating that the equivalence of the reduced linearizations implies the equivalence of the linearizations. But under the assumptions (G1)-(G5) the reduced linearizations are the zero tuples, see Remark 2.8. Therefore if $k = 1$ then under the assumptions (G2) and (G3) the linearizations are equivalent, and Theorem A1 follows from Theorem 6.1.

The proof of Theorem 6.1 requires the notion of the algebraic restriction introduced in [Z] and two theorems from [DJZ2], involving the algebraic restrictions, on the classification of varieties in a symplectic space.

Within this work we need only the definition of the zero algebraic restriction. Let θ be a germ at $0 \in \mathbb{R}^{2k}$ of a 2-form on $\mathbb{R}^k$ and let $N \subset \mathbb{R}^k$ be any subset. Recall from [DJZ2] that θ has zero algebraic restriction to N if there exist a 1-form α vanishing at any point of N and a 2-form β, also vanishing at any point of N, such that $\theta = d\alpha + \beta$. We will use the following statements from [DJZ2]:

Table 2. Normal forms

	Coord.	$S_1^{k,*}$	$S_2^{k,*}$	ω^*
$k=1$ (G2), (G3)	$x,y\in\mathbb{R}$ $u,v\in\mathbb{R}^{n-1}$	$y=0$ $u=0$ $v=0$	$x=0$ $u=0$ $v=0$	$dx\wedge dy+\sum_{i=1}^{n-1}du_i\wedge dv_i$
$k=2\le n$ $k=3\le n$ (G1) − (G5)	$x,y\in\mathbb{R}^k$ $u,v\in\mathbb{R}^{n-k}$	$y=0$ $u=0$ $v=0$	$x=0$ $u=0$ $v=0$	$\sum_{i=1}^{k}dx_i\wedge dy_i+$ $\sum_{i=1}^{n-k}du_i\wedge dv_i+$ $dx_1\wedge dx_2+\frac{dy_1\wedge dy_2}{\lambda_1}$
$2\le k\le n$ (G1) − (G6)	$x,y\in\mathbb{R}^k$ $u,v\in\mathbb{R}^{n-k}$	$y=0$ $u=0$ $v=0$	$x=0$ $u=0$ $v=0$	$\sum_{i=1}^{k}dx_i\wedge dy_i+$ $\sum_{i=1}^{n-k}du_i\wedge dv_i+$ $\sum_{i=1}^{s}dx_{2i-1}\wedge dx_{2i}+$ $\sum_{i=1}^{s}\frac{dy_{2i-1}\wedge dy_{2i}}{\lambda_i}$ $s=[k/2]$
$n<k\le 2n-4$ $(G1)-(G6)$	$x,y\in\mathbb{R}^{2n-k}$ $u,v\in\mathbb{R}^{k-n}$	$y=0$	$x=0$	$\sum_{i=1}^{2n-k}dx_i\wedge dy_i+$ $\sum_{i=1}^{k-n}du_i\wedge dv_i+$ $\sum_{i=1}^{s}dx_{2i-1}\wedge dx_{2i}+$ $\sum_{i=1}^{s}\frac{dy_{2i-1}\wedge dy_{2i}}{H_i(u,v)}$ $s=[(2n-k)/2]$
$n<k=2n-3$ $n<k=2n-2$ (G1) − (G5) (G7)	$x,y\in\mathbb{R}^{2n-k}$ $u,v\in\mathbb{R}^{k-n}$	$y=0$	$x=0$	$\sum_{i=1}^{2n-k}dx_i\wedge dy_i+$ $\sum_{i=1}^{k-n}du_i\wedge dv_i+$ $dx_1\wedge dx_2+\frac{dy_1\wedge dy_2}{\lambda_1}$
$k=2n-1$ (G1), (G3)	$x,y\in\mathbb{R}$ $u,v\in\mathbb{R}^{n-1}$	$y=0$	$x=0$	$dx\wedge dy+\sum_{i=1}^{n-1}du_i\wedge dv_i$

Theorem 6.2 ([DJZ2], section 2.7)**.** *Let ω and $\tilde\omega$ be germs at $0\in\mathbb{R}^{2n}$ of symplectic forms on $\mathbb{R}^{2n}$ such that the 2-form $\omega-\tilde\omega$ has zero algebraic restriction to a quasi-homogeneous variety $N\subset\mathbb{R}^{2n}$. Then there exists a local diffeomorphism of $\mathbb{R}^{2n}$ which preserves N pointwise and sends $\tilde\omega$ to ω.*

One of examples of a quasi-homogeneous variety is the germ at $0 \in \mathbb{R}^s$ of the union $S_1 \cup \cdots \cup S_r$ of smooth submanifolds of $\mathbb{R}^s$ satisfying the following condition:

$$\dim\ T_0S_1 + \cdots + \dim\ T_0S_r = \dim\ (T_0S_1 + \cdots + T_0S_r). \tag{6.2}$$

Proposition 6.3 ([DJZ2], section 7.1)**.** *Let $S_1, .., S_r$ be germs at $0 \in \mathbb{R}^k$ of smooth submanifolds of $\mathbb{R}^k$ satisfying (6.2). Let θ be a germ at $0 \in \mathbb{R}^k$ of a 2-form on $\mathbb{R}^{2k}$. If $\theta(0) = 0$ and $\theta|_{TS_i} = 0$, $i = 1, .., r$, then θ has zero algebraic restriction to the set $S_1 \cup \cdots \cup S_r$.*

Finally, the proof of Theorem 6.1 requires the following simple extension of the classical Darboux theorem.

Theorem 6.4 (simple extension of the classical Darboux theorem)**.** *Let ω and $\tilde{\omega}$ be germs at $0 \in \mathbb{R}^k$ of 2-forms on $\mathbb{R}^k$ of the maximal possible rank $2[k/2]$ and such that $\omega(0) = \tilde{\omega}(0)$. Then there exists a local diffeomorphism Ψ of $\mathbb{R}^k$ such that $\Psi^*\tilde{\omega} = \omega$ and $\Psi'(0) = id$.*

Proof of Theorem 6.1. By the assumption (G1) the 2-forms $\omega|_{TS_i^k}$ and $\tilde{\omega}|_{TS_i^k}$ have the maximal possible rank, $i = 1, 2$. By Theorem 6.4 there exist local diffeomorphisms $\Psi_i : (S_i^k, 0) \to (S_i^k, 0)$ such that $\Psi_i^*(\tilde{\omega}|_{TS_i^k}) = \omega|_{TS_i^k}$ and such that $\Psi_i'(0) = id$. Take a local diffeomorphism $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ preserving S_i^k such that $\Phi'(0) = id$ and $\Phi|_{S_i^k} = \Psi_i$. This diffeomorphism brings the second tuple in (6.1) to a tuple $(\mathbb{R}^{2n}, \hat{\omega}, S_1^k \cup S_2^k)$ where $\hat{\omega}(0) = \omega(0)$ and $\hat{\omega}|_{TS_i^k} = \omega|_{TS_i^k}$. We have showed that to prove Theorem 6.1 it suffices to prove it under the additional assumption that ω and $\tilde{\omega}$ have the same restrictions to the tangent bundle to the strata. Since $1 \le k \le n$ then the assumption (G2) implies (6.2) with $r = 2$ and by Proposition 6.3 the 2-form $\omega - \tilde{\omega}$ has zero restriction to the variety $S_1^k \cup S_2^k$. Now Theorem 6.1 follows from Theorem 6.2.

7. Proof of Theorems C and A2

In this section $n < k \le 2n - 1$. Like in section 6 we work with two tuples

$$\mathcal{T} = (\mathbb{R}^{2n}, \omega, S_1^k \cup S_2^k)_0, \quad \widetilde{\mathcal{T}} = (\mathbb{R}^{2n}, \widetilde{\omega}, S_1^k \cup S_2^k)_0. \tag{7.1}$$

Recall the notation $Q = S_1^k \cap S_2^k$. At first we generalize Propositions 4.1 and 4.2.

Proposition 7.1 (cf. Proposition 4.1)**.** *Assume that the tuples (7.1) satisfy* (G1)-(G4) *and have the same reduced linearization at any point*

$z \in Q$. Then there exists a local diffeomorphism Φ sending the tuple $\widetilde{T}$ to a tuple $\widehat{\mathcal{T}} = (\mathbb{R}^{2n}, \widehat{\omega}, S_1^k \cup S_2^k)_0$ such that $\mathcal{T}$ and $\widehat{\mathcal{T}}$ have the same linear approximation at any point $z \in Q$.

Proof. The proof is almost the same as that of Proposition 4.1. Assumptions (G1)-(G4) allow us to chose local coordinates in which Table 1 holds for the germ of the tuple $\mathcal{T}$ at any point $z \in Q$, and $\omega(z)$ has the form (4.1) where the matrices A, B, C depend smoothly on $z \in Q$. This normal form implies Proposition 7.1. $\square$

Proposition 7.2 (cf. Proposition 4.2)**.** *Assume that the tuples (7.1) satisfy* (G1)-(G6)*. If their reduced linearizations at any point $z \in Q$ have the same characteristic numbers then there exists a local diffeomorphism sending the tuple $\widetilde{T}$ to a tuple $\widehat{\mathcal{T}} = (\mathbb{R}^{2n}, \widehat{\omega}, S_1^k \cup S_2^k)_0$ such that $\mathcal{T}$ and $\widehat{\mathcal{T}}$ have the same reduced linear approximation at any point $z \in Q$.*

Proof. The proof repeats that of Proposition 4.2, the only difference is that now all matrices depend on the parameter $z \in Q$ and that at the end of the proof one should use Theorem 4.6 with families $(A(z), B(z))$ instead of individual couples (A, B). Since the number of distinct eigenvalues is maximal possible (by condition (G6), all transformations depend smoothly on $z \in Q$. $\square$

Theorems C and A2 follow from Propositions 4.1 and 4.2 and the following theorem generalizing the linearization Theorem 6.1.

Theorem 7.3 (cf. Theorem 6.1)**.** *If the tuples (7.1) satisfy* (G1)-(G3) *and have the same linearization at any point $z \in Q$ then they are equivalent.*

Theorem 7.3 is proved in the next section.

Proof of Theorem C. Take two tuples (7.1) satisfying (G1)-(G6). Take a coordinate system (x, y, z) on $\mathbb{R}^{2n}$ such that $S_1 = \{y = 0\}$ and $S_2 = \{x = 0\}$. By (G3) the restrictions of ω and $\tilde{\omega}$ to Q are symplectic. Assume that there exists a local diffeomorphism ϕ of Q that maps $\widetilde{\omega}|_{TQ}$ to $\omega|_{TQ}$ and the characteristic Hamiltonians of the tuple $\widetilde{\mathcal{T}}$ to the characteristic Hamiltonians of the tuple $\mathcal{T}$. Take a prolongation of ϕ to $\mathbb{R}^{2n}$ of the form $\Phi(x, y, z) = (x, y, \phi(z))$. Such Φ preserves the strata S_1 and S_2 and sends the tuple $\widetilde{\mathcal{T}}$ to a tuple $\mathcal{T}_1 = (\mathbb{R}^{2n}, \omega_1, S_1^k \cup S_2^k)_0$ where $\omega|_{TQ} = \omega_1|_{TQ}$. The reduced linearizations of the tuples $\mathcal{T}$ and $\mathcal{T}_1$ have the same characteristic numbers at any point $z \in Q$. Now Theorem C is a logical corollary of Theorem 7.3 and Propositions 7.1, 7.2. $\square$

Proof of Theorem A2. Let $k = 2n-1$. Take two tuples (7.1) satisfying (G2)and (G3). Note that for $k = 2n-1$ condition (G1) is always satisfied since the form ω is non-degenerate and conditions (G4) and (G5) follow from (G2) and (G3). In fact, for any point $z \in Q$ one has

$$(T_z S_i^{2n-1})^\omega = l_{i,z}, \quad l_{i,z} = ker\omega|_{T_z S_i^{2n-1}}, \quad i = 1, 2.$$

This equation also implies that the reduced linearizations of the tuples $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ at any point $z \in Q$ are the same zero tuple. Now Theorem A2 is a logical corollary of Theorem 7.3 and Proposition 7.1. □

8. Proof of Theorem 7.3

The assumption in Theorem 7.3 that the tuples (7.1) have the same linearization at any point $z \in Q$ means

$$\omega(z) = \widetilde{\omega}(z), \quad z \in Q. \tag{8.1}$$

Proposition 8.1. *Assume that the tuples (7.1) satisfy conditions* (G1)-(G3) *and condition (8.1). Then there exists of a local diffeomorphism* $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ *satisfying the following requirements:*

(a) Φ *preserves the strata* S_i^k, $i = 1, 2$*;*

(b) $\Phi(z) = z$ *and* $\Phi'(z) = id$ *for any* $z \in Q$*;*

(c) $\Phi|^*_{S_i^k}\left(\widetilde{\omega}|_{TS_i^k}\right) = \omega|_{TS_i^k}$, $i = 1, 2$.

This proposition allows us to prove Theorem 7.3 under the following additional assumption:

$$\widetilde{\omega}|_{TS_i^k} = \omega|_{TS_i^k}, \quad i = 1, 2. \tag{8.2}$$

Under this additional assumption Theorem 7.3 is a direct corollary of Theorem 6.2 and the following statement.

Proposition 8.2. *Let* $k > n$ *and let* S_1^k *and* S_2^k *be germs at the origin of* k*-dimensional smooth submanifolds of* $\mathbb{R}^{2n}$ *such that* $T_0S_1^k + T_0S_2^k = T_0\mathbb{R}^{2n}$*. Let* $Q = S_1^k \cap S_2^k$*. If* ω *and* $\tilde{\omega}$ *are germs of symplectic forms satisfying (8.1) and (8.2) then the form* $\theta = \tilde{\omega} - \omega$ *has zero algebraic restriction to* $S_1^k \cup S_2^k$*.*

We have reduced Theorem 7.3 to Propositions 8.1 and 8.2. The rest of this section is devoted to the proof of these propositions.

To prove Proposition 8.1 we need the following slight modification of the Darboux-Givental theorem.

Proposition 8.3. *Let $d = 2s$ or $d = 2s+1$. Let μ and $\tilde{\mu}$ be germs at the origin of closed 2-forms on $\mathbb{R}^d$ or of maximal rank $2s$ and let $Q \subset \mathbb{R}^d$ be a smooth submanifold. Assume that $\mu(z) = \tilde{\mu}(z)$ for any $z \in Q$. In the odd-dimensional case also assume that* $\ker \mu$ *and* $\ker \tilde{\mu}$ *are not tangent to Q. Then there exists a local diffeomorphism Φ of $\mathbb{R}^d$ which preserves Q pointwise, has identity linear approximation at any point $z \in Q$, and which brings $\tilde{\mu}$ to μ.*

Proof. In the even-dimensional case the required diffeomorphism can be constructed by the homotopy method exactly in the same way as in the proof of the Darboux-Givental' theorem. The odd-dimensional case reduces to the even-dimensional case as follows. Take a hypersurface H which contains Q and which is transversal to the kernels od ω and $\tilde{\omega}$. The restrictions of ω and $\tilde{\omega}$ to this hypersurface are symplectic. Take a local diffeomorphism $\widehat{\Phi}$ of H which preserves Q pointwise, has identity linear approximation at any point $z \in Q$, and which brings $\tilde{\mu}|_{TH}$ to $\mu|_{TH}$. Take vector fields X and $\tilde{X}$ which generate the kernels of μ and $\tilde{\mu}$ respectively and agree at any point of Q. The required local diffeomorphism Φ of $\mathbb{R}^d$ has the following form $\Phi(p) = (\Psi^{-s} \circ \widehat{\Phi} \circ \tilde{\Psi}^s)(p)$ for $p \in \mathbb{R}^d$, where Ψ^t and $\tilde{\Psi}^t$ are the flows of X and $\tilde{X}$ respectively and s is a real number such that $\tilde{\Psi}^s(p) \in H$. □

Proof of Proposition 8.1. The assumptions (G1)-(G3) imply the assumptions of Proposition 8.3 for the 2-forms $\mu = \omega|_{TS_1^k}$ and $\tilde{\mu} = \tilde{\omega}|_{TS_1^k}$ as well as for the 2-forms $\mu = \omega|_{TS_2^k}$ and $\tilde{\mu} = \tilde{\omega}|_{TS_2^k}$. Applying Proposition 8.3 to the first two restrictions we obtain a local diffeomorphism Ψ_1 of S_1^k which preserves Q pointwise, has identity linear approximation at any point $z \in Q$, and which brings $\tilde{\omega}|_{TS_1^k}$ to $\omega|_{TS_1^k}$. Applying Proposition 8.3 to the restrictions $\omega|_{TS_2^k}$ and $\tilde{\omega}|_{TS_2^k}$ we obtain a local diffeomorphism Ψ_2 of S_2^k which preserves Q pointwise, has identity linear approximation at any point $z \in Q$, and which brings $\tilde{\omega}|_{TS_2^k}$ to $\omega|_{TS_2^k}$. Take a local coordinate system (x, z, y) on $\mathbb{R}^{2n}$ such that $S_1^k = \{y = 0\}$ and $S_2^k = \{x = 0\}$ and construct the following prolongations of Ψ_1 and Ψ_2 to local diffeomorphisms of $\mathbb{R}^{2n}$:

$$\Phi_1(x, z, y) = (\Psi_1(x, z), y), \ \Phi_2(x, z, y) = (x, \Psi_2(z, y)).$$

Then $\Phi = \Phi_1 \circ \Phi_2$ has the required properties. □

Now we prove Proposition 8.2. We need the following lemma.

Lemma 8.4. *Under the notations and assumptions of Proposition 8.2 one has $\theta = d\alpha$ where α is a 1-form such that*

$$j_z^1 \alpha = 0, \quad \alpha|_{T_z S_i} = 0, \quad z \in Q, \quad i = 1, 2. \tag{8.3}$$

Proof. Choose a coordinate system (x, y, z) on $\mathbb{R}^{2n}$ such that $S_1 = \{y = 0\}$ and $S_2 = \{x = 0\}$. Consider the family of mappings

$$F_t(x, y, z) = (tx, ty, z),\ t \in [0; 1].$$

Let V_t is a vector field along F_t such that $V_t \circ F_t = F_t'$ (see [DJZ1] for details) and let

$$\alpha = \int_0^1 F_t^*(V_t \rfloor \theta) dt.$$

Then by (8.1) we have $F_0^*\theta = 0$ and it follows

$$\theta = F_1^*\theta - F_0^*\theta = \int_0^1 (F_t^*\theta)' dt = d\alpha.$$

Since F_t preserves the strata S_1 and S_2, it is easy to see that (8.1) and (8.2) imply (8.3). □

Proof of Proposition 8.2. We use the following statement which was proved in [DJZ2]: if S_1, S_2 and Q are as Proposition 8.2 and a 1-form α satisfies (8.3) then α has zero algebraic restriction to $S_1 \cup S_2$, i.e. $\alpha = \tilde{\alpha} + df$ where $\tilde{\alpha}$ is a 1-form vanishing at any point of $S_1 \cup S_2$ and f is a function vanishing at any point of $S_1 \cup S_2$. Proposition 8.2 is a direct corollary of this statement and Lemma 8.4. □

9. Appendix. Symplectic invariants of pairs of submanifolds of different dimensions

The results of this work can be generalized to tuples

$$(\mathbb{R}^{2n}, \omega, S_1^{k_1} \cup S_2^{k_2})_0 \tag{9.1}$$

where ω is a symplectic form on $\mathbb{R}^{2n}$ and $S_1^{k_1}$ and $S_2^{k_2}$ are smooth submanifolds of $\mathbb{R}^{2n}$ of different dimensions $k_1 < k_2$ such that $k_1 + k_2$ is an even number. Here, as above, $(\ )_0$ means that all objects are germs at the origin. Define the reduced linearization of (9.1) to be the tuple $(W, \sigma, U_1 \cup U_2)$, where

for k_1, k_2 even :

$$W = \begin{cases} (T_pS_1^{k_1} + T_pS_2^{k_2}) \cap (T_pS_1^{k_1} + (T_pS_2^{k_2})^\omega) & \text{if } k_1 + k_2 \le 2n \\ \left(T_pS_1^{k_1} \cap T_pS_2^{k_2}\right)^\omega \cap \left(T_pS_1^{k_1} + (T_pS_2^{k_2})^\omega\right) & \text{if } k_1 + k_2 > 2n; \end{cases}$$

$$\text{for } k_1, k_2 \text{ odd}:$$

$$W = \begin{cases} (T_pS_1^{k_1} + T_pS_2^{k_2}) \cap (T_pS_1^{k_1} + (T_pS_2^{k_2})^\omega) \cap (\ell_1 + \ell_2)^\omega \\ \text{if } k_1 + k_2 \leq 2n; \\ \left(T_pS_1^{k_1} \cap T_pS_2^{k_2}\right)^\omega \cap \left(T_pS_1^{k_1} + (T_pS_2^{k_2})^\omega\right) \cap (\ell_1 + \ell_2)^\omega \\ \text{if } k_1 + k_2 > 2n. \end{cases}$$

$$\sigma = \omega|_W, \quad U_1 = T_pS_1^{k_1} \cap W, \quad U_2 = T_pS_2^{k_2} \cap W.$$

Theorem 9.1. *Under the genericity assumptions* (G1') - (G5') *and* (G8) *listed below the following holds:*

(a) (W, σ) *is a symplectic space of dimension* $4s$ *and* U_1 *and* U_2 *are transversal* $2s$*-dimensional symplectic subspaces where*

$$s = s(k_1, k_2, 2n) = \min\{[k_1/2], [(2n - k_2)/2]\}.$$

(b) *Two tuples of the form (9.1) are equivalent if and only if their reduced linearizations are equivalent.*

The genericity assumptions (G1')-(G5') are obvious generalizations of the assumptions (G1)-(G5) for the case $k_1 < k_2$.

(**G1'**) the restriction of ω to the tangent bundle to the strata $S_1^{k_1}$ and $S_2^{k_2}$ has the maximal possible rank $2[k_1/2]$ and $2[k_2/2]$.

(**G2'**) The couple $(S_1^{k_1}, S_2^{k_2})_0$ is regular. This means that $T_0S_1^{k_1} \cap T_0S_2^{k_2} = \{0\}$ if $k_1 + k_2 \leq 2n$ and $T_0S_1^{k_1} + T_0S_2^{k_2} = T_0M^{2n}$ if $k_1 + k_2 > 2n$.

(**G3'**)) If $k_1 + k_2 \leq 2n$ then the restriction of ω to the space $T_0S_1^{k_1} + T_pS_2^{k_2}$ has maximal possible rank $k_1 + k_2$. If $k_1 + k_2 > 2n$ then the restriction of ω to the space $T_0S_1^{k_1} \cap T_0S_2^{k_2}$ has maximal possible rank $(k_1 + k_2 - 2n)$.

(**G4'**) if k_1, k_2 are odd then ω does not annihilate the 2-plane $(\ell_1 + \ell_2)$, where $\ell_1 = \ker \omega|_{T_0S_1^{k_1}}$, $\ell_2 = \ker \omega|_{T_0S_2^{k_2}}$

(**G5'**) the space $(T_0S_1^{k_1})^\omega$ is transversal to $T_0S_2^{k_2}$ in T_0M^{2n}.

The genericity assumption (G8) is "new"; it always holds in the case $k_1 = k_2$.

(**G8**) the restriction of ω to the space $(T_0S_1^{k_1})^\omega \cap T_0S_2^{k_2}$ has the maximal possible rank $k_2 - k_1$.

Note that if $k_1 = 1$ or $k_2 = 2n - 1$ then $s(k_1, k_2, 2n) = 0$. Therefore Theorem 9.1 implies the following statements generalizing Theorems A1 and A2.

Theorem A'.$k_1 = 1$ *or* $k_2 = 2n - 1$ *then all tuples (9.1) satisfying* (G1')-(G5') *and* (G8) *are equivalent.*

Theorem 9.1 allows us to define the characteristic numbers and characteristic Hamiltonians of tuples (9.1) exactly in the same way as we defined these invariants in Sections 2 and 3, for the case $k_1 = k_2$. Under the genericity assumptions

(**G6'**) If $s(k_1, k_2, 2n) \geq 2$ then the number of characteristic numbers is maximal possible, i.e. $s(k_1, k_2, 2n)$

one has the following theorems generalizing Theorems B and C:

Theorem B'. *Let* k_1 *and* k_2 *be integers of the same parity such that* $2 \leq k_1 \leq k_2 \leq 2n - 2$ *and* $k_1 + k_2 \leq 2n$. *In the problem of classifying tuples (9.1) satisfying* (G1')-(G6') *and* (G8) *the tuple of characteristic numbers is a complete invariant.*

Theorem C'. *Let* k_1 *and* k_2 *be integers of the same parity such that* $3 \leq k_1 \leq k_2 \leq 2n-2$ *and* $k_1 + k_2 > 2n$. *In the problem of classifying tuples (9.1) satisfying* (G1')-(G6') *and* (G8)*, a complete invariant is the tuple of characteristic Hamiltonians on the symplectic manifold* $S_1^{k_1} \cap S_2^{k_2}$ *defined up to a symplectomorphism of this manifold.*

For the case $k_1 = k_2$ Theorems 9.1, B', and C' are proved in sections 4 - 8. The proofs for the case $k_1 < k_2$ are almost the same.

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