Abstract convex analysis in metric spaces.

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1. Axiomatic definitions of convex sets and convex functions

In the theory of optimization of convex functions essential role is played by convex analysis. The basic notions concerning convex analysis are following

(i) a convex set,

(ii) a convex function.

There is natural question, what we ought to do when we consider optimization problems concerning of functions defined on a set, which is not convex, or even if the set is convex the defined function is not convex.

Mathematicians knows the answer on this problem. The natural way is to extend the definitions of the both considered notions. The extensions ought be relatively large, since in this case we have a lot of models. On the other hand it ought be sufficiently narrow, since we want to obtain some nontrivial results.

Thus we have a problem how we can define in an axiomatic way

(i) a convex set,

(ii) a convex function.

We shall start with axiomatic definition of convex sets. The first, I can call it a prehistorical step in the subject, was done by Kuratowski (1922), who define the closure operation in an axiomatic way. Let X be a set called later the space. Let $u(\cdot) : 2^X \to 2^X$ be an operation with the following property

$$A \subset u(A), \tag{1.1}$$

$$u(A) \subset u(B) \text{ if } A \subset B, \tag{1.2}$$

$$u(u(A)) = u(A), \tag{1.3}$$

$$u(A \cup B) = u(A) \cup u(B). \tag{1.4}$$

Then we say that u is a *closure operation* (see Kuratowski (1922)). Kuratowski showed that in this way we can define a topology on X. Using his notation $u(A) = \bar{A}$, we can define the closed sets as those sets A, that $A = \bar{A}$ and we can define open sets as those, which complement is closed.

Observe that if X is a linear space than the operation $co(\cdot)$ of taking of convex hull satisfies the axioms (1.1), (1.2), (1.3) but does not satisfy axiom (1.4). Thus it is natural to call an operation $u(\cdot)$ satisfying the axioms (1.1), (1.2), (1.3) *convex hull operation*. Since it does not lead to misunderstanding we shall denote convex hull operation by $co(\cdot)$. A set $A \subset X$ is called *convex*, if $co(A) = A$

Now we shall give an example of convex hull operation related to a notion of $\mathcal{M}$-convex sets. Let $\mathcal{M}$ be a family of subsets of a set X, we say that a set $A \subset X$ is *$\mathcal{M}$-convex*, if there is a subfamily $\mathcal{M}_1 \subset \mathcal{M}$, such that

$$A = \bigcap_{B \in \mathcal{M}_1} B. \tag{1.5}$$

The family of all $\mathcal{M}$-convex sets we shall denote by $\mathcal{M}_{\text{conv}}$. There are simple example showing that two different family of sets $\mathcal{M}$ and $\overline{\mathcal{M}}$ induce the same family of convex sets, i.e. $\mathcal{M}_{\text{conv}} = \overline{\mathcal{M}}_{\text{conv}}$ and there is a natural question about maximal and minimal families giving the same $\mathcal{M}$-convex sets. Such an maximal family always exists. It is unique and it is just $\mathcal{M}_{\text{conv}}$. Minimal families in general do not exist. I know only if a family $\mathcal{M}$ is finite, then always such a minimal family $\mathcal{M}_{\min}$ exists.

A very nice historical description of the notion of $\mathcal{M}$-convex sets is done in the paper Danzer, Grünbaum, Klee (1963) in §9.

Quite often we make an additional assumption about the family $\mathcal{M}$, namely we require

($\star$) $X \in \mathcal{M}$ and the intersection of sets belonging to any subfamily $\mathcal{M}_1 \subset \mathcal{M}$ is a member of $\mathcal{M}$.

In other words we request that $\mathcal{M} = \mathcal{M}_{\text{conv}}$. Ky Fan (Fan, K.) (1963) called the family satisfying ($\star$) *$\cap$-stable* and Soltan (1984) called them *convexity*.

Having the convexity we can introduce a notion of a *hull operation* (or *convexification*) $\text{conv}_{\mathcal{M}}(A)$ of an arbitrary set A, defined in the following way

$$\text{conv}_{\mathcal{M}}(A) = \bigcap \{M : M \in \mathcal{M}, A \subset \mathcal{M}\} \tag{1.6}$$

Putting $u(A) = \text{conv}_{\mathcal{M}}(A)$ we trivially obtain that $u(\cdot)$ satisfies axioms (1.1), (1.2), (1.3).

Observe that if $u(\cdot)$ is a hull operation than the family $\mathcal{M}_u = \{u(A) : A \subset 2^X\} = \{A : u(A) = A\}$ satisfies ($\star$). Indeed for any subfamily $\mathcal{M}_1 \subset \mathcal{M}_u$ we have

$$u(\bigcap_{A \in \mathcal{M}_1} A) \subset \bigcap_{A \in \mathcal{M}_1} u(A) \subset u(\bigcap_{A \in \mathcal{M}_1} u(A)) \subset u(\bigcap_{A \in \mathcal{M}_1} A).$$

Thus the both approaches by hull operation and by convexity are equivalent.

Now we shall try to define a convex function. It is obvious that the classical definition of the convex function is not very useful, since it uses in an essential way the linear structure of a domain of the function. However in the classical case we have the following observation. Let X be a Banach space. Let $\Omega \subset X$ be a convex set with non empty interior. Then a real-valued function $f(\cdot)$ is convex if and only if

$$f(x) = \sup\{\ell(x) + c : \ell \in X^*, c \in \mathbb{R}, \ell(\cdot) + c \leq f(\cdot)\}. \tag{1.7}$$

where X^* is the conjugate space and $\ell(\cdot) + c \leq f(\cdot)$ means that $\ell(x) + c \leq f(x)$ for all $x \in \Omega$.

Using this observation replacing the family of continuous linear functionals X^* by an arbitrary family of functions defined on X we obtain the following definition.

Let X be a set called later the space. Let Φ be an arbitrary family of functions defined on X and having values in the extended real line $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. For a function $f(x)$ defined on X and having values in the extended real line $\bar{\mathbb{R}}$ we shall denote by

$$f^{\Phi}(x) = \sup\{\phi(x) + c : \phi \in \Phi, c \in \mathbb{R}, \phi + c \leq f\}. \tag{1.8}$$

The function $f^{\Phi}(x)$ is called the Φ-*convexification* of the function f. If $f^{\Phi}(x) = f(x)$, i.e.,

$$f(x) = \sup\{\phi(x) + c : \phi \in \Phi, c \in \mathbb{R}, \phi + c \leq f\}. \tag{1.9}$$

we say that the function f is Φ-*convex*. The set of all Φ-convex functions we shall denote by Φ_{conv}. Identifying a function $f(\cdot)$ with its epigraph $E_f = \{(t,x) \in \mathbb{R} \times X : t \in \mathbb{R}, x =\in X, t \geq f(x)$, we obtain that on the set of functions the operation $(\cdot)^{\Phi}$ is convex hull operation.

There is a natural question how to define the hull operation on functions in an axiomatic way. Singer (2000) propose the following approach. We consider an operation v mapping the space $\bar{\mathbb{R}}^X$ of functions mapping a set X into $\bar{\mathbb{R}}$ into itself satisfying the following axioms

$$v(f) \leq f \tag{$\overline{1.1}$}$$

$$f_1 \leq f_2 \text{ implies } v(f_1) \leq v(f_2), \tag{$\overline{1.2}$}$$

$$v(v(f)) = v(f). \tag{$\overline{1.3}$}$$

(which correspond axioms (1.1), (1.2), (1.3) for epigraphs). The operation v we shall call *hull operation*.

Having a hull operation we can define a family of functions $\Phi_v = \{v(f) : f \in 2^X\} = \{f : v(f) = f\}$. It is easy to see that Φ_v-convexification is nothing else as v. Observe that the family Φ_v satisfies the following condition

($\bar{\star}$) for any subfamily $\Phi_1 \subset \Phi_v$ supremum of functions belonging to Φ_1 belongs to Φ_v.

Indeed

$$\sup_{\phi\in\Phi_1} \phi = \sup_{\phi\in\Phi_1} v(\phi) \le v(\sup_{\phi\in\Phi_1} \phi) \le \sup_{\phi\in\Phi_1} \phi.$$

There is a natural question how to formulate Kuratowski axiom (1.4). May be

$$v(\inf\{f,g\}) = \inf\{v(f), v(g)\}? \tag{$\overline{1.4}$}$$

Is is interesting to find a non-trivial example of such hull operation.

Similar as in the case of $\mathcal{M}$-convex sets in the case of Φ-convex functions we have the problem of the existence of minimal (in the sense of inclusion) families Φ^0 such that $\Phi^0_{\text{conv}} = \Phi_{\text{conv}}$. I know the following two examples

1. Let X be a Banach space and let Φ be the class of all convex functions defined on X. Let X^* be the conjugate space. Then X^* the unique minimal (in the sense of inclusion) family such that ${X^*}_{\text{conv}} = \Phi$.
2. Let $X = \mathbb{R}^n$. Let $\mathbf{H}$ be the set of all harmonic entire functions. Then $\mathbf{H}$ the unique minimal (in the sense of inclusion) family generating $\mathbf{H}_{\text{conv}}$.

In the proof of those examples we are using the fact that an affine (or harmonic) function defined on the whole space and bounded from below (or above) is constant.

In the case when X is an open convex set smaller than the whole space, in both cases such minimal family does not exist.

2. Φ-subgradients and Φ-subdifferentials.

Having the notions of Φ-convex functions we can in a natural way to define Φ-subgradients and Φ-subdifferentials. Let $f(\cdot)$ be a real-valued function defined on X. Similarly as in the classical case, a function $\phi(\cdot) \in \Phi$ will be called a Φ-*subgradient* of the function $f(\cdot)$ at a point x_0 if

$$f(x) - f(x_0) \ge \phi(x) - \phi(x_0) \tag{2.1}$$

for all $x \in X$. The set of all Φ-subgradients of the function $f(\cdot)$ at a point x_0 we shall call the Φ-*subdifferential* of the function f at the point x_0 and we shall denote it by $\partial_\Phi f\big|_{x_0}$.

Of course $\partial_\Phi f\big|_{(\cdot)}$ is a multifunction mapping X into 2^Φ. It is not too difficult to observe that this multifunction is cyclic monotone, i.e. for arbitrary n and $x_0, x_1, ..., x_n = x_0 \in X$ and $\phi_{x_i} \in \partial_\Phi f\big|_{x_i}$, $i = 0, 1, 2, ..., n$, we have

$$\sum_{i=1}^{n} [\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_{i-1}}(x_i)] \ge 0. \tag{2.2}$$

Unfortunately, it need not to be maximal cyclic monotone, as in the classical case. However, generalizing the Rockafellar theorem (1970b) we can show that every maximal cyclic monotone multifunction $\Gamma(\cdot)$ is a subdifferential of a certain Φ-convex function $f(\cdot)$

(Rolewicz (1996), Pallaschke and Rolewicz (1997), see also Levin (1999)). Then the following questions arises.

Problem 2.1. Is the function $f(\cdot)$ determined uniquely up to the constant, as it is in the classical case?

On this very abstract formalism using only the order in the space of real numbers we can generalize duality theory (Moreau (1963), (1966), (1970), Kutateladze and Rubinov (1971), (1972), (1976), Elster and Nehse (1974), Dolecki and Kurcyusz (1978) and many others)

Also it is possible to develop the Lagrange theory ((Kurcyusz (1976), Balder (1977), Dolecki and Kurcyusz (1978) and many others).

3. Localization and globalization.

Let (X, d_X) be a metric space. In this case we can localize the notions introduced in the first two sections.

Let $\mathcal{M}$ be a family of subsets of a the space X. We say that a set $A \subset X$ is *locally $\mathcal{M}$-convex*, if for each $x_0 \in A$, and each neighbourhood U of x_0 there is a neighbourhood $V \subset U$ of x_0, such that the intersection $A \cap V$ is $\mathcal{M}$-convex.

Of course each $\mathcal{M}$-convex set is locally $\mathcal{M}$-convex. Thus a following question arises, when a locally $\mathcal{M}$-convex set is also $\mathcal{M}$-convex. We say that a class $\mathcal{F}$ of sets has $\mathcal{M}$-globalization property if each locally $\mathcal{M}$-convex set is also $\mathcal{M}$-convex. In the case when $\mathcal{M}$ has $\mathcal{M}$-globalization property we say briefly that $\mathcal{M}$ has globalization property. Is is not difficult to see that in arbitrary metric space (or even more general topological space) the classes of closed sets and open sets have globalization property.

The nice example of localization and globalization is following. Let (X, d_X) be a linear metric locally convex space. Take as $\mathcal{M}$ the family of convex sets. We can say that a set $A \subset X$ is *locally $\mathcal{M}$-convex* (briefly *locally convex*), if for each $x_0 \in A$, there is a convex neighbourhood V of x_0, such that the intersection $A \cap V$ is $\mathcal{M}$-convex (i.e. simply convex). Of course, in this case $\mathcal{M}$ does not have globalization property. Indeed, each open set is locally convex. However, each closed connected locally convex is convex (see Tietze (1928), Matsumura (1928) in $\mathbb{R}^n$ and Klee (1951) in general case). Thus the class $\mathcal{F}$ of closed connected locally convex has $\mathcal{M}$-globalization property.

If we have a linear structure in the space we can uniformed the localization procedure. We say that a set $A \subset X$ is *uniformly locally convex* if there is a neighbourhood V of 0 such that for each $x \in X$ the set $A \cap (x + V)$ is convex. Of course, each uniformly locally convex set is also locally convex. The converse is not true. For uniformly locally convex sets we have

Proposition 3.1 (Rolewicz (2000)). *Let X be a locally convex topological space. Let $A \subset X$ be a uniformly locally convex set. Then its closure $\bar{A}$ is uniformly locally convex.*

Proposition 3.1 together with Tietze-Matsumura-Klee theorem implies

Proposition 3.2 *Let X be a locally convex topological space. Let $A \subset X$ be a connected uniformly locally convex set. Then its closure $\bar{A}$ is convex.*

Similarly as in the case of sets we can consider localization and globalization in the case of functions.

Let Φ be a family of continuous functions defined on X. We say that a real-valued function $f(\cdot)$ is *locally Φ-convex*, if for each $x_0 \in \mathrm{dom} f$, there is a neighbourhood V of x_0, such that the function $f\big|_V$ if $\Phi\big|_V$-convex. The set of all locally Φ-convex functions we shall denote by $\Phi^{loc}_{\mathrm{conv}}$. We say that a function $\phi \in \Phi$ is a *local Φ-subgradient* of the function $f(\cdot)$ at a point x_0, if there is a neighbourhood V of x_0, such that the function $\phi\big|_V$ is a $\Phi\big|_V$- subgradient of the function $f(\cdot)$ at the point x_0. The set of all local Φ-subgradients of the function $f(\cdot)$ at a point x_0 we shall call the *local Φ-subdifferential* of the function f at the point x_0 and we shall denote it by $\partial^{loc}_{\Phi} f\big|_{x_0}$. If $\partial^{loc}_{\Phi} f\big|_{x_0} \neq \emptyset$ for all $x_0 \in X$ we say that the function $f(\cdot)$ is *locally Φ-subdifferentiable.*

Having those definitions we have the following globalization problems. Under which conditions putted on X and Φ following implications hold

(i) each locally Φ-convex function is Φ-convex,

(ii) each local Φ-subgradient can be extended to a Φ-subgradient,

(iii) each locally Φ-subdifferentiable function is a Φ-subdifferentiable function.

Of course if implication (ii) holds, then the implication (iii) also holds. The converse is not true as follows from the following

Example 3.3. Let $X = [-1, 1]$. Let $\Phi = \{-c|x - x_0| : 0 \leq c \leq 1, -1 \leq x_0 \leq 1\}$. It is easy to see that a function $f(\cdot)$ is Φ-convex if and only if it is a Lipschitz function with a constant non-greater than 1 and that each such function is also Φ-subdifferentiable. In this example the implications (i) and (iii) holds but the implication (ii) does not hold. Indeed, let $f(x) = \min[0, |x + \frac{1}{2}|]$. The function $\phi(x) \equiv 0$ is a local Φ-subgradient of the function $f(\cdot)$ at the point 0, but it is not a Φ-subgradient of the function $f(\cdot)$ at the point 0.

We do not know to much about the relations between implications (i) (resp. (ii), (iii)). We know only

Proposition 3.4. (Rolewicz (1995b)) *Let X be an open set of a normed space E. Let Φ be the set of linear continuous functionals restricted to X. Then implication (i) (resp. (ii), (iii)) holds if and only if X is convex.*

There is a natural question how is the situation when X is a one-connected open set in $\mathbb{R}^n$ and Φ is the class of harmonic functions defined on X.

4. Differentiability

Another natural consequence of introducing a metric is a possibility to introduce a Fréchet differentiability.

We shall say that a function $f(\cdot)$ mapping a metric space (X, d_X) into $\mathbb{R}$ is *Fréchet Φ-differentiable* at a point x_0 if there is a function $\phi \in \Phi$ such that

$$\lim_{x \to x_0} \frac{|[f(x) - f(x_0)] - [\phi(x) - \phi(x_0)]|}{d_X(x, x_0)} = 0. \tag{4.1}$$

The function ϕ will be called a *Fréchet Φ-gradient* of the function $f(\cdot)$ at the point x_0. The set of all Fréchet Φ-gradients of the function $f(\cdot)$ at the point x_0 is called *Fréchet Φ-differential* of the function $f(\cdot)$ at the point x_0 and it is denoted by $\partial^F_\Phi f\big|_{x_0}$.

In general a Φ-subdifferentiable function may not be Fréchet Φ-differentiable at any point. Indeed

Example 4.1. Let $X = \mathbb{R}$ and let $\Phi = \{\phi(x) = -|x - x_0|, x_0 \in \mathbb{R}\}$. It is easy to see that a function $f(\cdot)$ is Φ-convex if and only if it is a Lipschitz function with the Lipschitz constant non-greater than 1. Thus the function $f(x) \equiv 0$ is Φ-subdifferentiable. It is easy to see that it is not Fréchet Φ-differentiable at any point.

However under proper assumptions we can obtain an extension of the famous Asplund theorem to the case of metric spaces.

The assumptions are as follow:

(a) Φ is an additive group,

(sL) Φ is a set of Lipschitz functions. Moreover the space ${}^{\Phi}\!/\!_{\mathbb{R}}$ is separable in the Lipschitz norm $\|\phi\|_L$,

(wm) the family Φ has the *weak k-monotonicity property*, $0 < k \le 1$, i.e. for all $x \in X$, $\phi \in \Phi$ and $t > 0$, there is a $y \in X$ such that $0 < d_X(x, y) < t$ and

$$|\phi(y) - \phi(x)| \ge k\|\phi\|_L d_X(y, x). \tag{4.2}$$

Theorem 4.2 (Rolewicz (2002)). *Let X be a metric space. Let Φ be a family of Lipschitz functions satisfying assumptions* (a), (sL) *and* (wm). *Let a multifunction Γ mapping X into 2^Φ be monotone and such that* dom $\Gamma = X$ (*i.e.*, $\Gamma(x) \neq \emptyset$ *for all* $x \in X$). *Then there exists a residual set Ω such that Γ is single-valued and continuous* (*i.e. simultaneously lower semi-continuous and upper semi-continuous*) *at each point of Ω.*

Recall that in the case of normed spaces Gateaux differentiability of a convex continuous functions $f(\cdot)$ at a point x is equivalent to the fact that the subdifferential $\partial f\big|_x$ consists of one point only. Moreover the continuity of Gateaux differentials in the norm operator topology implies that these differentials are the Fréchet differential. Similarly we have an extension of this fact to metric spaces (Rolewicz (1995), (1996)). As a consequence we get

Theorem 4.3 (Rolewicz (2002)). *Let X be a metric space, which is of the second category on itself* (*in particular, let X be a complete metric space*). *Let Φ be a family of Lipschitz functions satisfying assumptions* (a), (sL) *and* (wm). *Let $f(\cdot)$ be a continuous*

Φ-subdifferentiable function. Then there is a residual set Ω such that the function $f(\cdot)$ is Fréchet Φ-differentiable at every point $x_0 \in \Omega$. Moreover, on Ω the Fréchet Φ-gradient is unique and it is continuous in the metric d_L.

In the previous papers (Rolewicz (1994), (1995), (1995b), (1999)) and in the book Pallaschke - Rolewicz (1997) the assumption (wm) was formulated in the stronger way namely that

(m) the family Φ has the *k-monotonicity property*, $0 < k \leq 1$, i.e. for all $x \in X$, $\phi \in \Phi$ and $t > 0$, there is a $y \in X$ such that $0 < d_X(x, y) < t$ and

$$\phi(y) - \phi(x) \geq k\|\phi\|_L d_X(y, x). \tag{4.3}$$

Observe that if X is a compact set, then condition (m) is never satisfied, but the condition (wm) can hold (Rolewicz (1999b)).

We say that the family Φ has the *strict k-monotonicity property*, $0 < k \leq 1$, if for all $x \in X$ and all $\phi \in \Phi$, there is a constant $r_{x,\phi}$ such that for all $0 < s < r_{x,\phi}$ there is $y \in X$ such that $d_X(x, y) = s$ and

$$\phi(y) - \phi(x) \geq k\|\phi\|_L d_X(y, x). \tag{4.3_1}$$

The strict k-monotonicity property is related to the notion of κ-super-metric coupling introduced by Penot (2002).

Of course if the family Φ has the *strict k-monotonicity property*, then it has the k-monotonicity property. We do not know if the converse implication holds.

We know that the answer is positive in the case of open sets $X \subset \mathbb{R}^n$ and Φ consisting of continuously differentiable functions and for families Φ consisting of Lipschitz functions defined on open interval $(a, b) \subset \mathbb{R}$ (Rolewicz (2003b)) .

There is also a similar problem concerning weak k-monotonicity property. We shall say that a family Φ has *weak strict k-monotonicity property*, $0 < k \leq 1$, if for all $x \in X$, $\phi \in \Phi$ there is a constant $r_{x,\phi}$ such that for all $0 < s < r_{x,\phi}$ there is $y \in X$ such that $d_X(x, y) = s$ and

$$|\phi(y) - \phi(x)| \geq k\|\phi\|_L d_X(y, x). \tag{$4.3'_1$}$$

In this case the following example shows that the answer is negative.

Example 4.4. Let $X = [0, 1]$. Let $\Phi = \{c\phi : c \in \mathbb{R}\}$, where

$$\phi(x) = \inf_n 4|x - \frac{1}{2^n}|.$$

It is easy to see that ϕ is a Lipschitz function with constant 4. Take $x = 0$. By simple calculation we obtain that ϕ has weak $\frac{1}{3}$-monotonic property, but is does not have weak strict k-monotonic property for any $k > 0$. Of course on the set $X' = (0, 1]$ Φ has weak strict k-monotonic property for arbitrary k, $0 < k \leq 1$.

As a consequence of Theorem 4.3 we obtain a weak version of the famous Asplund theorem

Corollary 4.5. *Let X be a convex set in a separable Banach space E. Suppose that Int $X \neq \emptyset$. Let $f(\cdot)$ be a convex function defined on X. Suppose that the dual space E^* is separable. Then there is a residual set $\Omega \subset X$ such that the function $f(\cdot)$ is Fréchet Φ-differentiable at every point $x_0 \in \Omega$. Moreover, on Ω the Fréchet Φ-gradient is unique and it is continuous in the norm in conjugate space E^*.*

Proof. Let Φ be restrictions of E^* to X. Of course the conditions (a), (sL) and (wm) are satisfied. The proof consists of two parts. In the first part we are proving that $f(\cdot)$ is Φ-subdifferentiable. It is obvious consequence of Hahn-Banach theorem. Then by Theorem 4.3 we obtain our result. □

5. Uniformly approximate Φ-subdifferentials

There is a natural problem how to define a Φ-subdifferential in the case of non-Φ-convex functions.

The natural approach is following. Let $f(\cdot)$ be a real-valued function defined on X. Similarly as in the classical case (compare for example Fabian (1989), Ioffe (1983), (1984), (1986), (1989), (1990), (2000), Mordukchovich (1980), (1988)), a function $\phi(\cdot) \in \Phi$ will be called a *approximate Φ-subgradient* of the function $f(x)$ at a point x_0 if

$$\liminf_{x \to x_0} \frac{[f(x) - f(x_0)] - [\phi(x) - \phi(x_0)]}{d_X(x, x_0)} \geq 0. \tag{5.1}$$

The set of all Φ-subgradients of the function $f(\cdot)$ at a point x_0 we shall call *approximate Φ-subdifferential* of the function f at the point x_0 and we shall denote it by $\partial^w_\Phi f\big|_{x_0}$.

Of course $\partial^w_\Phi f\big|_{(\cdot)}$ is a multifunction mapping a domain of $\partial^w_\Phi f\big|_{(\cdot)}$ into 2^Φ.

Observe that (5.1) holds if and only there is a non-negative non-decreasing function $\beta_{x_0}(\cdot)$ defined on the interval $[0, +\infty)$ and such that $\lim_{u \downarrow 0} \beta_{x_0}(u) = 0$ and

$$\frac{[f(x) - f(x_0)] - [\phi(x) - \phi(x_0)]}{d_X(x, x_0)} \geq -\beta_{x_0}(d_X(x, x_0)). \tag{5.2}$$

Indeed, the function

$$\beta_{x_0}(s)) = \sup_{\{x : d_X(x, x_0) \leq s\}} \Big| \frac{[f(x) - f(x_0) - \phi(x) - \phi(x_0)]}{d_X(x, x_0)} \Big| \tag{5.3}$$

has the requested property.

Putting $\alpha_{x_0}(u) = u\beta_{x_0}(u)$ we can rewrite (5.2) in the form

$$f(x) - f(x_0) \geq \phi(x) - \phi(x_0) - \alpha_{x_0}(d_X(x, x_0)). \tag{5.4}$$

Unfortunately $\beta_{x_0}(\cdot)$ (and thus $\alpha_{x_0}(\cdot)$) can be different in each point and we are not able to use this definition for the problem of differentiation on a residual set. Thus there is an idea of a uniformization of this notion.

Let (X, d_X) be a metric space. Let $\alpha(t)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that

$$\lim_{t \downarrow 0} \frac{\alpha(t)}{t} = 0. \tag{5.5}$$

Let $f(\cdot)$ be a real-valued function defined on X. Let $x_0 \in X$. A function $\phi_{x_0} \in \Phi$ such that

$$f(x) - f(x_0) \geq \phi_{x_0}(x) - \phi_{x_0}(x_0) - \alpha(d_X(x, x_0)). \tag{5.6}$$

we shall call a *uniform approximate* Φ*-subgradient of the function* $f(\cdot)$ *at* x_0 *with the modulus* $\alpha(\cdot)$ (or briefly $\alpha(\cdot)$-Φ-subgradient of the function $f(\cdot)$ at x_0). The set of all $\alpha(\cdot)$-Φ-subgradient of the function $f(\cdot)$ at x_0 will be called the $\alpha(\cdot)$-Φ*-subdifferential with of the function* $f(\cdot)$ *at* x_0 and it will be denoted by $\partial_\Phi^{-\alpha} f\big|_{x_0}$.

We say that a function $f(x)$ is $\alpha(\cdot)$-Φ*-subdifferentiable* if $\partial_\Phi^{-\alpha} f\big|_{x_0} \neq \emptyset$ for all $x_0 \in X$.

In a similar way we can consider a uniform Fréchet differential. It seems that this notion was not investigated also in the classical case. In this classical case uniform Fréchet differentiability is equivalent to continuous differentiability, provided that we consider functions on relatively regular domains (see Rolewicz (2003)).

Let, as before, $\alpha(t)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that (5.5) holds. We say that a multifunction Γ mapping X into 2^Φ is $\alpha(\cdot)$*-monotone* if for all $\phi_x \in \Gamma(x)$, $\phi_y \in \Gamma(y)$ we have

$$\phi_x(x) + \phi_y(y) - \phi_x(y) - \phi_y(x) + \alpha(d_X(x, y)) \geq 0. \tag{5.7}$$

It is easy to see that, the subdifferential $\partial_\Phi^\alpha f\big|_x$ of a $\alpha(\cdot)$-Φ-subdifferentiable function is a $2\alpha(\cdot)$- monotone multifunction of x. Adapting the method of Preiss and Zajiček (1984) and the proof of Rolewicz (1994) (see also proof of Theorem 2.4.11 of Pallaschke - Rolewicz (1997)) we can obtain

Theorem 5.1 (Rolewicz (2002)). *Let* X *be a metric space. Let* Φ *be a family of Lipschitz functions satisfying assumptions* (a), (sL) *and* (wm). *Let a multifunction* Γ *mapping* X *into* 2^Φ *be* $\alpha(\cdot)$*-monotone and such that* dom $\Gamma = X$ (*i.e.,* $\Gamma(x) \neq \emptyset$ *for all* $x \in X$). *Then there exists a residual set* Ω *such that* Γ *is single-valued and continuous on the set* Ω.

Since the subdifferential $\partial_\Phi^\alpha f\big|_x$ of a $\alpha(\cdot)$-Φ-differentiable function is a $2\alpha(\cdot)$- monotone multifunction of x, we immediately obtain

Corollary 5.2 (Rolewicz (2002)). *Let* X *be a metric space, which is of the second category on itself* (*in particular, let* X *be a complete metric space*).. *Let* Φ *be a family* *of Lipschitz functions satisfying assumptions* (a), (sL) *and* (wm). *Let* $f(\cdot)$ *be an* $\alpha(\cdot)$-Φ*-subdifferentiable function. residual set* Ω *such that the* $\alpha(\cdot)$-Φ*-subdifferential* $\partial^{\alpha}_{\Phi} f\big|_x$ *is single-valued and continuous in the metric* d_L.

Similarly, as in the previous section we can prove

Proposition 5.3 (Rolewicz (1999)). *Let* X *be a metric space. Let* Φ *be a family of Lipschitz functions defined on* X *satisfying* (a) . *Let* $f(\cdot)$ *be an* $\alpha(\cdot)$-Φ*-subdifferentiable function. If the subdifferential* $\partial^{\alpha}_{\Phi} f\big|_x$ *is lower semi-continuous at* x_0 *in the Lipschitz norm, then it is the Fréchet* Φ*-differential of the function* $f(\cdot)$ *at the point* x_0, *and of course it is lower semi-continuous at* x_0 *in the Lipschitz norm, too.*

Let X be a metric space, which is of the second category on itself. Let Ω_0 be a residual set in X. Let Ω be a residual set in Ω_0. Then trivially Ω is a residual set in X. Thus as a consequence of Theorem 5.1 and Proposition 5.3 we obtain

Theorem 5.4 (compare Rolewicz (1999)). *Let* X *be a metric space, which is of the second category on itself* (*in particular, let* X *be a complete metric space*). *Let* Φ *be a family of Lipschitz functions satisfying assumptions* (a), (sL) *and* (wm). *Let* $f(\cdot)$ *be a continuous* $\alpha(\cdot)$-Φ*-subdifferentiable function. Then there is a residual set* Ω *such that the function* $f(\cdot)$ *is Fréchet* Φ*-differentiable at every point* $x_0 \in \Omega$. *Moreover, on* Ω *the Fréchet* Φ*-gradient is unique and it is continuous in the metric* d_L.

Suppose that X is an open set of a Banach space Y having separable dual Y^*. Let Φ be the family of continuous linear functionals restricted to X. It is easy to see that Φ satisfies assumptions (a), (sL) and (wm). Thus Theorem 5.4 can be rewritten in this case in the following way

Theorem 5.4_B. *Let* X *be an open set of a Banach space* Y *having separable dual* Y^*. *Let* Φ *be the family of linear continuous functionals restricted to* X, $\Phi = Y^*\big|_X$. *Let* $f(\cdot)$ *be a continuous* $\alpha(\cdot)$-Φ*-subdifferentiable function. Then there is a residual set* Ω *such that the function* $f(\cdot)$ *is Fréchet differentiable at every point* $x_0 \in \Omega$. *Moreover, on* Ω *the Fréchet* Φ*-gradient is unique and it is continuous in the conjugate norm* $\|\cdot\|^*$.

6. $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions.

Theorem 5.4_B have a certain disadvantage. Namely it is difficult to check that $f(\cdot)$ is a continuous $\alpha(\cdot)$-Φ-subdifferentiable function. Thus it is a natural question how to describe the class of functions which have this property. In this section $(X, \|\cdot\|)$ will be a normed space and X^* will be its dual. Let $\Omega \subset X$. By Φ we shall denote the restriction to Ω the elements of X^*. Since it does not lead to misunderstanding, in this section we shall omit Φ in the definitions of $\alpha(\cdot)$-Φ-subdifferentiability and $\alpha(\cdot)$-Φ-monotonicity.

Let $\alpha(t)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that (5.5) holds. Let $(X, \|\cdot\|)$ be a normed space. Let Ω be a convex subset of X. Let $f(\cdot)$ be a real valued function defined on Ω. We say that the function $f(\cdot)$ is *$\alpha(\cdot)$-paraconvex* with a constant* $C > 0$ if for all $x, y \in \Omega$ and $0 \le t \le 1$

* In general in the definition of $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions

$$f\big(tx+(1-t)y\big) \leq tf(x)+(1-t)f(y)+C\alpha(\|x-y\|). \tag{6.1}$$

We say that the function $f(\cdot)$ is *$\alpha(\cdot)$-paraconvex*, if there is a constant $C>0$ such that the function $f(\cdot)$ is $\alpha(\cdot)$-paraconvex with the constant $C>0$. For $\alpha(t)=t^2$ this definition was introduced in Rolewicz (1979) and the t^2-paraconvex functions were called simply paraconvex functions. In Rolewicz (1979b) the notion was extended to the case $\alpha(t)=t^\gamma, 1\leq\gamma\leq 2$, and in the t^γ-paraconvex functions were called γ-paraconvex functions.

We say that the function $f(\cdot)$ is *strongly $\alpha(\cdot)$-paraconvex with a constant $C_1>0$* if for all $x,y\in\Omega$ and $0\leq t\leq 1$

$$f\big(tx+(1-t)y\big) \leq tf(x)+(1-t)f(y)+C_1\min[t,(1-t)]\alpha(\|x-y\|). \tag{6.2}$$

If there is a constant $C_1>0$ such that the function $f(\cdot)$ is strongly $\alpha(\cdot)$-paraconvex with the constant $C_1>0$, we say that the function $f(\cdot)$ is *strongly $\alpha(\cdot)$-paraconvex* .

Of course every function $f(\cdot)$ strongly $\alpha(\cdot)$-paraconvex with a constant $C_1>0$ is also $\alpha(\cdot)$-paraconvex with the constant $C_1>0$. The converse is not true (Rolewicz (2000)).

It was shown in Rolewicz (1979, 1979b) that for $\alpha(t)=t^\gamma, 1<\gamma\leq 2$, any $\alpha(\cdot)$-paraconvex function is simultaneously strongly $\alpha(\cdot)$-paraconvex.

There are $\alpha(\cdot)$-paraconvex functions $f(\cdot):X\to\mathbb{R}$, which are not strongly $\alpha(\cdot)$-paraconvex. Conditions warranting that each $\alpha(\cdot)$-paraconvex functions is automatically strongly $\alpha(\cdot)$-paraconvex a reader can find in the paper Rolewicz (2000).

Proposition 6.1 (Rolewicz (2002)) *Let $(X,\|\cdot\|)$ be a normed space. Let Ω be a convex set in X. Let $f(\cdot)$ be a real valued function defined on Ω. If the function $f(\cdot)$ is $\alpha(\cdot)$-subdifferentiable, then it is $\alpha(\cdot)$-paraconvex with constant 1. If additionally*

$$\alpha(ts)\leq t\alpha(s), \tag{6.3}$$

for $0<t<1$ and $s>0$ then the function $f(\cdot)$ is strongly $\alpha(\cdot)$-paraconvex with constant 2.

For the purpose of further considerations we shall localize the notions of $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions.

We say that a real-valued function $f(\cdot)$ defined on a locally convex set $\Omega\subset X$ is *locally (strongly) $\alpha(\cdot)$-paraconvex with a constant $C>0$* if for each $x_0\in\Omega$ there is a neighbourhood U of x_0 such that the function $f(\cdot)$ restricted to the set U (strongly) $\alpha(\cdot)$-paraconvex with the constant $C>0$. We say that a function $f(\cdot)$ is *locally (strongly)*

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the assumption (5.5) is replaced by the weaker assumption

$$\limsup_{t\downarrow 0}\frac{\alpha(t)}{t}<+\infty,$$

but for our considerations the stronger assumption (5.5) will be more adequate.

$\alpha(\cdot)$*-paraconvex* if there is $C > 0$, such that it is locally (strongly) $\alpha(\cdot)$-paraconvex with the constant C.

We say that a real-valued function $f(\cdot)$ defined on a uniformly locally convex set $\Omega \subset X$ is *uniformly locally* (*strongly*) $\alpha(\cdot)$*-paraconvex with a constant* $C > 0$ if there is a neighbourhood V of 0 such that for each $x_0 \in \Omega$ the function $f(\cdot)$ restricted to the set $\Omega \cap (x_0 + V)$ is (strongly) $\alpha(\cdot)$-paraconvex with the constant $C > 0$. If there is a constant $C > 0$, such that the function $f(\cdot)$ is uniformly locally (strongly) $\alpha(\cdot)$-paraconvex with the constant C we say that the function $f(\cdot)$ is *uniformly locally* (*strongly*) $\alpha(\cdot)$*-paraconvex*.

In general a local $\alpha(\cdot)$-Φ-subgradient need not to be an $\alpha(\cdot)$-Φ-subgradient. However in the case of strongly $\alpha(\cdot)$-paraconvex functions, defined on open convex sets we have

Theorem 6.2 (Rolewicz (2000), in the case of $\alpha(t) = t^\gamma$ Jourani (1996)). *Let* $(X, \|.\|)$ *be a real Banach space, which has the separable dual* X^*. *Let* $f(\cdot)$ *be a strongly* $\alpha(\cdot)$*-paraconvex function with constant* 1, *defined on an open convex subset* $\Omega \subset X$. *Then each local* $\alpha(\cdot)$*-subgradient of the function* $f(\cdot)$ *at a point* x_0 *is automatically an* $\alpha(\cdot)$*-subgradient of the function* $f(\cdot)$ *at the point* x_0.

Thus we have

Proposition 6.3. *Let* $(X, \|.\|)$ *be a real Banach space, which has the separable dual* X^*. *Let* $f(\cdot)$ *be a locally strongly* $\alpha(\cdot)$*-paraconvex function, defined on an open subset* $\Omega \subset X$. *Then the local* $\alpha(\cdot)$*-*Φ*-subdifferential of function* $f(\cdot)$, $\partial_\Phi^{\alpha,\text{loc}} f\big|_x$, *is a locally* $2\alpha(\cdot)$*-monotone multifunction.*

Proof. Let $x_0 \in \Omega$. Since the function $f(\cdot)$ is a locally strongly $\alpha(\cdot)$-paraconvex function, there is a convex open neighbourhood U of x_0 such that the function $f\big|_U(x)$ is strongly $\alpha(\cdot)$-paraconvex function. Thus by Theorem 6.2 $\partial_\Phi^{\alpha,\text{loc}} f\big|_U = \partial_\Phi^{\alpha,} f\big|_U$. Hence by Proposition 3.1 $\partial_\Phi^{\alpha,\text{loc}} f\big|_U$ is an $2\alpha(\cdot)$-monotone multifunction. Therefore $\partial_\Phi^{\alpha,} f\big|_U$ is a locally $2\alpha(\cdot)$-monotone multifunction. □

As a simple consequence of Proposition 6.1 and 6.3 we get

Proposition 6.4. *Let* $(X, \|\cdot\|)$ *be a normed space. Let* Ω *be a* (*uniformly*) *locally convex set in* X. *Let* $f(\cdot)$ *be a real valued function defined on* Ω. *If the function* $f(\cdot)$ *is* $\alpha(\cdot)$*-subdifferentiable, then it is* (*uniformly*) *locally* $\alpha(\cdot)$*-paraconvex with constant 1. If additionally*

$$\alpha(ts) \leq t\alpha(s), \tag{6.3}$$

for $0 < t < 1$ *and* $s > 0$ *then the function* $f(\cdot)$ *is* (*uniformly*) *locally strongly* $\alpha(\cdot)$*-paraconvex with constant* 2.

Let $f(\cdot)$ be a real valued function defined on a uniformly locally convex set Ω. If for arbitrary $\varepsilon > 0$ there is a $\delta > 0$ such that

$$tf(x) + (1-t)f(y) - f\big(tx + (1-t)y\big) \geq -\varepsilon t(1-t)\|x - y\| \tag{6.10}$$

for arbitrary $x, y \in \Omega$ such that $\|x-y\| \leq \delta$ we say that the function $f(\cdot)$ *uniformly approximate convex* (Rolewicz (2001b)). This is a uniformization of the notion of approximate convex functions introduced by Luc, Ngai and Théra (Luc-Ngai-Théra (2000)).

Corollary 6.5 (Rolewicz (2001b)). *Let $(X, \|\cdot\|)$ be a normed space. Let Ω be a uniformly locally convex set in X. Let $f(\cdot)$ be a real valued function defined on Ω. Suppose that (5.5) holds. If the function $f(\cdot)$ is $\alpha(\cdot)$-subdifferentiable, then it is uniformly approximate convex.*

Between uniformly approximate convex and strongly $\alpha(\cdot)$-paraconvex functions the following relations are true.

Proposition 6.6 (Rolewicz 2001b)). *Let $(X, \|\cdot\|)$ be a normed space. Let Ω be a uniformly locally convex set in X. Let $f(\cdot)$ be a real valued function defined on Ω. If the function $f(\cdot)$ is uniformly locally strongly $\alpha(\cdot)$-paraconvex, then $f(\cdot)$ is a uniformly approximate convex function.*

Conversely, if $f(\cdot)$ is a uniformly approximate convex function, then there is a non-decreasing function $\alpha(\cdot)$ mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ satisfying (5.5) such that the function $f(\cdot)$ is uniformly locally strongly $\alpha(\cdot)$-paraconvex.

The reversing of Propositions 6.1 requests the openness of the set Ω.

We start with

Proposition 6.7 (cf. Rolewicz (2000)). *Let $(X, \|\cdot\|)$ be a normed space. Let a real-valued function $f(\cdot)$ defined on a (locally) convex set $\Omega \subset X$ be (locally) strongly $\alpha(\cdot)$-paraconvex. If the function $f(\cdot)$ is locally bounded, then it is locally Lipschitz.*

Using the category methods we can obtain

Proposition 6.8 (cf. Rolewicz (2000)). *Let $(X, \|\cdot\|)$ be a Banach space. Let a real-valued function $f(\cdot)$ defined on an open (locally) convex set $\Omega \subset X$ be (locally) strongly $\alpha(\cdot)$-paraconvex. Then it is locally Lipschitz.*

Proposition 6.9 (cf. Rolewicz (2001), in the case of $\alpha(t) = t^\gamma$ Jourani (1996)). *Let $f(\cdot)$ be a locally strongly $\alpha(\cdot)$-paraconvex function defined on an open set Ω of a Banach space X. Then the local $\alpha(\cdot)$-subdifferentials and Clarke subdifferentials coincide.*

Corollary 6.10 (cf. Rolewicz (2001)). *Let $f(\cdot)$ be a (locally) strongly $\alpha(\cdot)$-paraconvex function defined on an open (locally) convex set Ω of a Banach space X. Then $f(\cdot)$ is (locally) $\alpha(\cdot)$-subdifferentiable.*

Proof. By Proposition 6.8 $f(\cdot)$ is locally Lipschitz. Thus in each point its Clarke subdifferential is not empty. Thus by Proposition 6.9 its (local) $\alpha(\cdot)$-subdifferential is not empty, too. □

Since every open set is locally convex, we do not need to assume convexity of Ω in the local versions of Propositions 6.8, 6.9 and Corollary 6.10.

As a consequence of Corollary 6.10 and Theorem 5.8_B we get the following extension of the Asplund (1968) theorem

Theorem 6.11. *Let $(X, \|.\|)$ be a real Banach space, which has the separable dual X^*. Let $f(\cdot)$ be a locally strongly $\alpha(\cdot)$-paraconvex function, defined on an open subset $\Omega \subset X$. Then there is a subset A_f of the first category such that on the set $\Omega \setminus A_f$ the* *function $f(\cdot)$ is Fréchet differentiable. Moreover the Fréchet Φ-gradient is continuous in the conjugate norm $\|\cdot\|^*$.*

As a consequence of Corollary 6.10 and Proposition 6.6 we get the following

Theorem 6.12. *Let $(X, \|.\|)$ be a real Banach space, which has the separable dual X^*. Suppose that*

$$\alpha(ts) \leq t\alpha(s), \tag{6.3}$$

for $0 < s < 1$ and $t > 0$. Let $f(\cdot)$ be a locally strongly $\alpha(\cdot)$-paraconvex function, defined on an open convex subset $\Omega \subset X$. Then it is strongly $\alpha(\cdot)$-paraconvex.

Proof. By Corollary 6.10 the function $f(\cdot)$ is locally $\alpha(\cdot)$-subdifferentiable. Then by Theorem 6.11 it is $\alpha(\cdot)$-subdifferentiable. Thus by Proposition 6.1 it is strongly $\alpha(\cdot)$-paraconvex function. □

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