THE K-THEORY TYPE OF QUANTIZED CW-COMPLEXES

As the result of quantization, the topology of classical spaces transforms into the functional analysis of C*-algebras. The fact that K-theory makes sense in both realms is the main justification of the paradigm of noncommutative topology [9, 4]. A fundamental question here is how classical topological structures emerge as the classical limit of C*-algebraic structures. It is a highly nontrivial problem due to the singular behavior of quantized (co)homological invariants in the classical limit. For the analysis of such phenomena, we propose a rigorous framework of *cw-Waldhausen categories*, partially inspired by [11, 8], in which K-equivalences play the role of weak equivalences. The main idea is that classical topological constructions, or structures, in K-theory should be understood as classical limits of systems of separate quantizations of their parts in possibly non-isomorphic C*-algebras related by K-equivalences.

The most interesting case for such analysis is the CW-complex structure allowing computations in K-theory based on the Mayer–Vietoris principle. Since the topological gluing of compact spaces has its C*-algebraic counterpart in the pullbacks of unital C*-algebras [7], the natural class of quantizations is determined by choosing appropriate quantizations of the cells and gluing maps relaxed by inverting K-equivalences. As a quite nontrivial example, we construct a K-quantization of the Atiyah–Todd structure of the K-theory of complex projective spaces [3, 1], which is induced by the filtration by skeleta (coming from the standard system of hyperlane embeddings) and by the module structure over the representation ring $R(U(1))$ (coming from vector bundles associated with the Hopf fibration).

Now, it is important to note that arguments of Atiyah and Todd are based on the ring structure of K-theory, the multiplicative Chern character, integral cohomology rings, homotopy classes of classifying maps, contracting projective hyperplanes to obtain even-dimensional spheres ($\mathbb{CP}^n/\mathbb{CP}^{n-1} \cong S^{2n}$), and the Atiyah–Hirzebruch spectral sequence. Regretfully, none of these tools is available in noncommutative geometry, so we were forced to invent different methods.

For starters, recall that the K-theory of the multipullback quantum complex projective spaces [5, 6] has been computed by Albert Sheu in terms of Toeplitz projections [10]. The main difficulty in relating his result with the classical result of Atiyah and Todd is that the Toeplitz projections do not admit a naive classical limit. We overcome this difficulty by direct K-theoretic calculations based on presenting the needed C*-algebras as groupoid C*-algebras. Thus, we obtain a noncommutative counterpart of the Atiyah–Todd result providing the lacking topological and geometric understanding to the operator-algebraic work of Sheu. Furhermore, for the multipullback quantum complex projective plane, we relate the projections of Sheu (built from Toeplitz operators) with quantized Milnor-type idempotents (coming from the clutching construction) by deriving an explicit homotopy.

To end with, let us note that our guiding principle here is the Atiyah–Jänich theorem [2] which states that, for a compact Hausdorff space X, its even topological K-theory can be computed as the group of homotopy classes of mappings from X to the space $\mathcal{F}$ of Fredholm operators on a separable Hilbert space: $K^0(X) = [X, \mathcal{F}]$. The fact that $\mathbb{CP}^n$ can be built combinatorially from contractible pieces means that the right-hand side can be represented combinatorially with the help of Fredholm operators. Since each quantum disc carries a Toeplitz operator being a Fredholm operator of index -1, one can expect that this combinatorial Fredholm presentation for the multipullback quantum complex projective spaces will be built from Toeplitz operators. Better still, our strategy can be, roughly speaking, subsumed in the following diagram:

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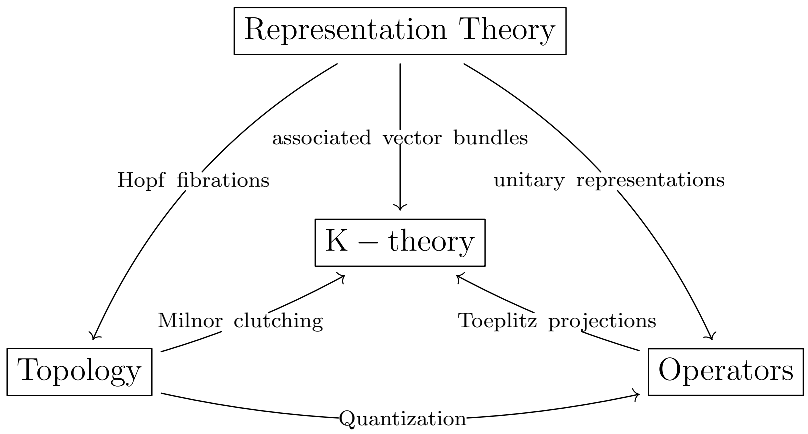

The upper part of the diagram makes sense both in the noncommutative and classical topology, whereas the lower part of the diagram can be understood as transforming topological information into purely operator-algebraic data by means of a quantum deformation.

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