Krzysztof Bogdan
Wrocław University of Technology

Heat kernel of isotropic nonlocal operators

I will discuss joint results with Tomasz Grzywny and Michał Ryznar [1, 3, 2]. Let $d = 1, 2, \ldots$. Let $\nu \colon [0, \infty) \to (0, \infty]$ be nonincreasing and denote $\nu(z) = \nu(|z|)$ for $z \in \mathbb{R}^d$. In particular, $\nu(z) = \nu(-z)$. We assume that $\int_{\mathbb{R}^d} \nu(z)dz = \infty$ and

$$\int_{\mathbb{R}^d} \left(|z|^2 \wedge 1\right) \nu(z)dz < \infty.$$

Summarizing, ν is a strictly positive density function of an isotropic infinite unimodal Lévy measure on $\mathbb{R}^d$. In short – ν is *unimodal*. For $x \in \mathbb{R}^d$ and $u : \mathbb{R}^d \to \mathbb{R}$,

$$\begin{aligned} Lu(x) &:= \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} (u(y) - u(x))\nu(y - x)\, dy \\ &= \tfrac{1}{2} \int_{\mathbb{R}^d} (u(x + z) + u(x - z) - 2u(x))\nu(z)\, dz. \end{aligned}$$

The limit exists, e.g., for $u \in C_c^\infty(\mathbb{R}^d)$, the smooth functions with compact support. Note that L is a non-local symmetric translation-invariant linear operator on $C_c^\infty(\mathbb{R}^d)$ with positive maximum principle. For example, if $0 < \alpha < 2$ and

$$\nu(z) = \frac{2^\alpha \Gamma((d + \alpha)/2)}{\pi^{d/2} |\Gamma(-\alpha/2)|} |z|^{-d-\alpha}, \quad z \in \mathbb{R}^d,$$

then L is the fractional Laplacian, denoted by $\Delta^{\alpha/2}$. In general, L is the generator of a Markovian semigroup–the transition semigroup of the isotropic pure-jump Lévy processes $\{X_t, t \geq 0\}$ with the Lévy measure ν. Thus, X is a càdlàg stochastic process with law $\mathbb{P}$, such that $X(0) = 0$ almost surely, the increments of X are independent, with radially nonincreasing density function $p_t(x)$ on $\mathbb{R}^d$, and the following Lévy-Khintchine formula holds for $\xi \in \mathbb{R}^d$:

$$\mathbb{E} e^{i\langle \xi, X_t\rangle} = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p_t(x)dx = e^{-t\psi(\xi)}, \quad \text{where } \psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos\langle \xi, x\rangle)\, \nu(dx).$$

Here $\mathbb{E}$ is the integration with respect to $\mathbb{P}$. We note that the Lévy-Khintchine exponent ψ is radial with the radial profile $\psi(\theta) = \psi(\xi)$ for $\xi \in \mathbb{R}^d$, $|\xi| = \theta$.

We estimate the *heat kernel* $p_D(t, x, y)$, $t > 0$, $x, y \in D$, of smooth open sets $D \subset \mathbb{R}^d$ for the operator L. Our estimates have a form of explicit factorization involving the transition density $p_t(x)$ of the Lévy process on the whole of $\mathbb{R}^d$, and the *survival probability* $\mathbb{P}^x(\tau_D > t)$, where $\tau_D = \inf\{t > 0 : X_t \notin D\}$ is the time of the first exit of X_t from D. Here $x \in \mathbb{R}^d$ and $\mathbb{P}^x$ is the law of $x + X$. For instance, if the radial profile of ψ has the so-called *global lower and upper scalings*, and D is a C^2 halfspace-like open set, then

$$p_D(t,x,y) \approx \mathbb{P}^x(\tau_D > t)\, p(t,x,y)\, \mathbb{P}^y(\tau_D > t), \qquad t > 0,\ x,y \in D.$$

Here,

$$p_t(x) \approx \psi^{-1}(1/t)^d \wedge \frac{t\,\psi(1/|x|)}{|x|^d},$$

$$\mathbb{P}^x(\tau_D > t) \approx 1 \vee (\psi(1/\delta_D(x))t)^{-1/2},$$

and $\delta_D(x) = \mathrm{dist}(x, D^c)$. The scaling conditions are understood as follows.[1] We say that ψ satisfies the weak *lower* scaling condition at infinity (WLSC) if there are numbers $\underline{\alpha} > 0$, $\underline{\theta} \geq 0$ and $\underline{c} \in (0,1]$, such that

$$\psi(\lambda\theta) \geq \underline{c}\lambda^{\underline{\alpha}}\psi(\theta) \quad \text{for} \quad \lambda \geq 1, \quad \theta > \underline{\theta}.$$

We write $\psi \in \mathrm{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ or $\psi \in \mathrm{WLSC}$. If $\psi \in \mathrm{WLSC}(\underline{\alpha}, 0, \underline{c})$, then we say that ψ satisfies the global WLSC. The weak *upper* scaling condition at infinity (WUSC) means that there are numbers $\overline{\alpha} < 2$, $\overline{\theta} \geq 0$ and $\overline{C} \in [1,\infty)$ such that

$$\psi(\lambda\theta) \leq \overline{C}\lambda^{\overline{\alpha}}\psi(\theta) \quad \text{for} \quad \lambda \geq 1, \quad \theta > \overline{\theta}.$$

In short, $\psi \in \mathrm{WUSC}(\overline{\alpha}, \overline{\theta}, \overline{C})$ or $\psi \in \mathrm{WUSC}$. Global WUSC means $\mathrm{WUSC}(\overline{\alpha}, 0, \overline{C})$. Explicit estimates are also given for bounded smooth sets D, etc.

References

[1] Krzysztof Bogdan, Tomasz Grzywny, and Michał Ryznar, *Density and tails of unimodal convolution semigroups*, J. Funct. Anal. **266** (2014), no. 6, 3543–3571. MR 3165234

[2] ______, *Dirichlet heat kernel for unimodal Lévy processes*, Stochastic Process. Appl. **124** (2014), no. 11, 3612–3650. MR 3249349

[3] ______, *Barriers, exit time and survival probability for unimodal Lévy processes*, Probab. Theory Related Fields **162** (2015), no. 1-2, 155–198. MR 3350043

[1] The notation that follows is a bit heavy, but it quantifies monotonicity properties of $\psi(\lambda)/\lambda^{\alpha}$.