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A new approach to the L^p theory of $-\Delta + b\cdot\nabla$ and its application to Feller processes with general drifts

I will discuss joint results with Yu. A. Semenov (Toronto). In $\mathbb{R}^d$, $d \geqslant 3$, consider the following classes of vector fields:

(1) We say that a $b:\mathbb{R}^d \to \mathbb{C}^d$ belongs to the Kato class $\mathbf{K}^{d+1}_\delta$, and write $b \in \mathbf{K}^{d+1}_\delta$, if $|b| \in L^1_{\rm loc}$ and there exists $\lambda = \lambda_\delta > 0$ such that

$$\|b(\lambda-\Delta)^{-\frac{1}{2}}\|_{1\to 1} \leqslant \delta.$$

(2) We say that a $b:\mathbb{R}^d \to \mathbb{C}^d$ belongs to $\mathbf{F}_\delta$, the class of form-bounded vector fields, and write $b \in \mathbf{F}_\delta$, if $|b| \in L^2_{\rm loc}$ and there exists $\lambda = \lambda_\delta > 0$ such that

$$\|b(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2} \leqslant \sqrt{\delta}.$$

(3) We say that a $b:\mathbb{R}^d \to \mathbb{C}^d$ belongs to $\mathbf{F}^{\frac{1}{2}}_\delta$, the class of *weakly* form-bounded vector fields, and write $b \in \mathbf{F}^{1/2}_\delta$, if $|b| \in L^1_{\rm loc}$ and there exists $\lambda = \lambda_\delta > 0$ such that

$$\||b|^{\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{4}}\|_{2\to 2} \leqslant \sqrt{\delta}.$$

The classes $\mathbf{F}_\delta$, $\mathbf{K}^{d+1}_\delta$ cover singularities of b of critical order, at isolated points or along hypersurfaces, respectively. Both classes have been thoroughly studied in the literature: after 1996, the Kato class $\mathbf{K}^{d+1}_\delta$, with $\delta > 0$ sufficiently small (yet allowed to be non-zero), has been recognized as 'the right' class for the Gaussian upper and lower bounds on the fundamental solution of $\partial_t - \Delta + b\cdot\nabla$ which, in turn, allow to construct an associated Feller semigroup (in C_b). The class $\mathbf{F}_\delta$, $\delta < 4$, is responsible for dissipativity of $\Delta - b\cdot\nabla$ in L^p, $p \geqslant \frac{2}{2-\sqrt{\delta}}$, needed to run an iterative procedure taking $p \to \infty$ (assuming additionally $\delta < \min\{4/(d-2)^2, 1\}$), which produces an associated Feller semigroup in C_∞; see [1] for details. We emphasize that, in general, the Gaussian bounds are not valid if $|b| \in L^d$ ($\subsetneq \mathbf{F}_0 := \bigcap_{\delta>0}\mathbf{F}_\delta$), while $b \in \mathbf{K}^{d+1}_0$ ($:= \bigcap_{\delta>0}\mathbf{K}^{d+1}_\delta$), in general, destroys L^p-dissipativity.

The class $\mathbf{F}^{1/2}_\delta$ contains both classes $\mathbf{F}_\delta$ and $\mathbf{K}^{d+1}_\delta$:

$$\mathbf{K}^{d+1}_\delta \subsetneq \mathbf{F}^{1/2}_\delta, \qquad \mathbf{F}_{\delta_1} \subsetneq \mathbf{F}^{1/2}_\delta \quad \text{for } \delta = \sqrt{\delta_1},$$

$$\left(b \in \mathbf{F}_{\delta_1} \text{ and } \mathsf{f} \in \mathbf{K}^{d+1}_{\delta_2}\right) \Longrightarrow \left(b + \mathsf{f} \in \mathbf{F}^{1/2}_\delta,\ \sqrt{\delta} = \sqrt[4]{\delta_1} + \sqrt{\delta_2}\right)$$

To deal with such general class of vector fields we will use ideas of E. Hille and J. Lions (alternatively, ideas of E. Hille and H.F. Trotter) to construct the generator $-\Lambda \equiv -\Lambda(b)$ (an operator realization of $\Delta - b\cdot\nabla$) of a quasi bounded holomorphic semigroup in L^2. This operator has some remarkable properties. Namely,

$$
\begin{tikzcd}
 & & \mathbf{F}_\delta^{1/2} & \\
 & \mathbf{K}_\delta^{d+1} \arrow[ur] & \mathbf{F}_{\delta^2} \arrow[u] & \\
\mathbf{K}_0^{d+1} \arrow[ur] & & \mathbf{F}_0 \arrow[u] & {[L^{d,\infty}+L^\infty]^d} \arrow[ul, "*"'] \\
 & & {[L^d+L^\infty]^d} \arrow[u] \arrow[ur] & \\
 & & {[L^p+L^\infty]^d \ (p>d)} \arrow[u] \arrow[uull] &
\end{tikzcd}
$$

General classes of vector fields $b:\mathbb{R}^d\to\mathbb{R}^d$ studied in the literature in connection with operator $-\Delta+b\cdot\nabla$. Here $\rightarrow$ stands for strict inclusion, and $\overset{*}{\rightarrow}$ reads "if $b=b_1+b_2\in[L^{d,\infty}+L^\infty]^d$, then $b\in\mathbf{F}_{\delta^2}$ with $\delta>0$ determined by the value of the $L^{d,\infty}$-norm of $|b_1|$,

$\rho(-\Lambda)\supset\mathcal{O}:=\{\zeta\mid\operatorname{Re}\zeta>\lambda\}$, and, for every $\zeta\in\mathcal{O}$,

$$
\begin{aligned}
(\zeta+\Lambda)^{-1} =& J_\zeta^3(1+H_\zeta^*S_\zeta))^{-1}J_\zeta\\
=& J_\zeta^4-J_\zeta^3H_\zeta^*(1+S_\zeta H_\zeta^*)^{-1}S_\zeta J_\zeta;\\
\|H_\zeta^*S_\zeta\| \le& \delta, \quad \|(\zeta+\Lambda)^{-1}\|_{2\to2}\le|\zeta|^{-1}(1-\delta)^{-1};\\
\|e^{-t\Lambda_r}\|_{r\to q} \le& c\,e^{t\lambda}t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})}, \quad 2\le r<q\le\infty;
\end{aligned}
$$

where $J_\zeta:=(\zeta-\Delta)^{-\frac14},\ H_\zeta:=|b|^{\frac12}J_{\bar\zeta},\ S:=b^{\frac12}\cdot\nabla J_\zeta^3,\ b^{\frac12}:=|b|^{-\frac12}b.$

As in the case $b\in\mathbf{F}_\delta$, it is reasonable to expect that there exists a strong dependence between the value of δ (effectively playing the role of a "coupling constant" for $b\cdot\nabla$) and smoothness of the solutions to the equation $(\zeta+\Lambda_r)u=f,\ \zeta\in\rho(-\Lambda_r),\ f\in L^r$ ($-\Lambda_r\equiv-\Lambda_r(b)$ is an operator realization of $\Delta-b\cdot\nabla$). Such a dependence does exist. Set

$$
m_d:=\pi^{\frac12}(2e)^{-\frac12}d^{\frac{d}{2}}(d-1)^{-\frac{d-1}{2}}, \quad \kappa_d:=\frac{d}{d-1}, \quad r_\mp:=\frac{2}{1\pm\sqrt{1-m_d\delta}}.
$$

It will be established that if $b\in\mathbf{F}_\delta^{1/2}$ and $m_d\delta<1$, then $(e^{-t\Lambda_r(b)},\ r\in[2,\infty[)$ extends by continuity to a quasi bounded C_0 semigroup in L^r for all $r\in]r_-,\infty[$. For every $r\in I_s:=]r_-,r_+[$, the semigroup is holomorphic, the resolvent set $\rho(-\Lambda_r(b))$

contains the half-plane $\mathcal{O} := \{\zeta \in \mathbb{C} \mid \operatorname{Re} \zeta > \kappa_d \lambda_\delta\}$, and the resolvent admits the representation

$$(\zeta + \Lambda_r(b))^{-1} = (\zeta - \Delta)^{-1} - Q_r(1 + T_r)^{-1} G_r, \quad \zeta \in \mathcal{O}, \tag{$\star$}$$

where Q_r, G_r, T_r are bounded linear operators on L^r; $D(\Lambda_r(b)) \subset W^{1+\frac{1}{q},r}$ $(q > r)$.

In particular, for $m_d \delta < 4\frac{d-2}{(d-1)^2}$, there exists $r \in I_s,\ r > d-1$, such that $(\zeta + \Lambda_r(b))^{-1} L^r \subset C^{0,\gamma}, \gamma < 1 - \frac{d-1}{r}$.

The results above yield the following: Let $b \in \mathbf{F}_\delta^{1/2}$ for some δ such that $m_d \delta < 4\frac{d-2}{(d-1)^2}$. Let $\{b_n\}$ be any sequence of bounded smooth vector fields, such that $b_n \to b$ strongly in $L^1_{\rm loc}$, and, for a given $\varepsilon > 0$ and some $\delta_1 \in]\delta, \delta + \varepsilon]$, $\{b_n\} \subset \mathbf{F}_{\delta_1}^{1/2}$. Then

$$s\text{-}C_\infty\text{-}\lim_{n \uparrow \infty} e^{-t\Lambda_{C_\infty}(b_n)} \tag{$\star\star$}$$

exists uniformly in $t \in [0,1]$, and hence determines a Feller semigroup $e^{-t\Lambda_{C_\infty}(b)}$.

The results ($\star$), ($\star\star$) can be obtained via direct investigation in L^r of the operator-valued function $\Theta_r(\zeta, b)$ defined by the right hand side of ($\star\star$) without appealing to L^2 theory (but again appealing to the ideas of E. Hille and H. F. Trotter). See [1] for details.

References

[1] Kinzebulatov D. and Semenov Yu. A. *On the theory of the kolmogorov operator in the spaces l^p and c_∞. i*, Preprint, arXiv:1709.08598 (2017).