C_0-semi-groups associated with evolutionary equations.

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Examples

We consider different types of diff. eq. on Hilbert space H:

- Let $M_0, M_1 \in L(H)$ and $A : \mathrm{dom}(A) \subseteq H \to H$ densely defined, closed linear.

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    $\rightsquigarrow$ well-posedness, if $-M_0^{-1}(M_1 + A)$ generates C_0-sg
  - If $\ker(M_0) \neq \{0\} \rightsquigarrow$ Differential-algebraic equation. What are admissible initial values?

Examples cont.

- Let $M_0, M_1 \in L(H)$ and $A : \mathrm{dom}(A) \subseteq H \to H$ dd, closed, linear.

$$(\partial_t M_0 + M_1 \tau_h + A)u = 0 \text{ on }]0, \infty[$$
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What are admissible histrories?

Common framework: Evolutionary equations.

1. Motivation: Initial values and histories
2. Evolutionary equations
3. Initial value problems for evolutionary equations

The setting

Definition

Let $\nu \geq 0$ and H HS. Define

$$L_{2,\nu}(\mathbb{R};H) := \{f : \mathbb{R} \to H\,;\ f \text{ meas.}, \int_{\mathbb{R}} \|f(t)\|^2 \exp(-2\nu t)\,\mathrm{d}t < \infty\}$$

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Moreover, define

$$\partial_{t,\nu} : H^1_\nu(\mathbb{R}; H) \subseteq L_{2,\nu}(\mathbb{R}; H) \to L_{2,\nu}(\mathbb{R}; H),\ u \mapsto u'$$

with

$$H^1_\nu(\mathbb{R}; H) := \{u \in L_{2,\nu}(\mathbb{R}; H)\,;\, u' \in L_{2,\nu}(\mathbb{R}; H) \text{ as distribution}\}.$$

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Remark

If $\nu > 0$, then $0 \in \rho(\partial_{t,\nu})$ with $\|\partial_{t,\nu}^{-1}\| = \frac{1}{\nu}$ and

$$\partial_{t,\nu}^{-1} f = \int_{-\infty}^{(\cdot)} f(s)\,\mathrm{d}s.$$

Definition

For $\varphi \in C_c(\mathbb{R}; H)$ define

$$(\mathcal{L}_\nu \varphi)(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\nu)s} \varphi(s)\, \mathrm{d}s \quad (t \in \mathbb{R}).$$

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$\mathcal{L}_\nu$ extends to unitary operator $\mathcal{L}_\nu : L_{2,\nu}(\mathbb{R}; H) \to L_2(\mathbb{R}; H)$. Moreover

$$\partial_{t,\nu} = \mathcal{L}_\nu^* (\mathrm{i}\,\mathrm{m} + \nu) \mathcal{L}_\nu,$$

where $\mathrm{m}\, f = (t \mapsto t f(t))$ *with*

$$\mathrm{dom}(\mathrm{m}) \coloneqq \{ f \in L_2(\mathbb{R}; H)\,;\, (t \mapsto t f(t)) \in L_2(\mathbb{R}; H) \}.$$

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$$M(\mathrm{i}\,\mathrm{m} + \nu) : L_2(\mathbb{R}; H) \to L_2(\mathbb{R}; H),\ u \mapsto (t \mapsto M(\mathrm{i}t + \nu)u(t))$$

as well as

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Let $\nu > \nu_0$. Then $M(\partial_{t,\nu})$ is causal, that is

$$\operatorname{spt} u \subseteq \mathbb{R}_{\geq a} \Rightarrow \operatorname{spt} M(\partial_{t,\nu})u \subseteq \mathbb{R}_{\geq a}.$$

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$$\mathrm{spt}\, u \subseteq \mathbb{R}_{\geq a} \Rightarrow \mathrm{spt}\, M(\partial_{t,\nu})u \subseteq \mathbb{R}_{\geq a}.$$

Moreover, for $u \in L_{2,\nu} \cap L_{2,\mu}$:

$$M(\partial_{t,\nu})u = M(\partial_{t,\mu})u.$$

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Let $M : \mathbb{C}_{\mathrm{Re}>\nu_0} \to L(H)$ analytic, bdd and $A : \mathrm{dom}(A) \subseteq H \to H$ dd, closed, linear.

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Assume well-posedness conditions and consider

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Evolutionary eq. for v:

$$(\partial_{t,\nu}M(\partial_{t,\nu}) + A)v = \delta_0 x - \mathbb{1}_{\mathbb{R}_{>0}}\partial_{t,\nu}M(\partial_{t,\nu})g$$

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Definition

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$$\mathrm{His}_\nu := \{g\,;\, \exists x \in H : \mathcal{S}_\nu(\delta_0 x - \mathbb{1}_{\mathbb{R}_{>0}} \partial_{t,\nu} M(\partial_{t,\nu}) g) + g \in H^1_\nu\}$$

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Remark

If $g\in\mathrm{His}_\nu$, then x is uniquely determined and given by

$$\begin{aligned}x&=(M(\partial_{t,\nu})\mathbb{1}_{\mathbb{R}_{>0}}g(0-))(0+)\\&=(M(\partial_{t,\nu})g)(0-)-(M(\partial_{t,\nu})g)(0+)\\&=:\Gamma g.\end{aligned}$$

For $g \in \mathrm{His}_\nu$ set $v := \mathcal{S}_\nu(\delta_0 x - \mathbb{1}_{\mathbb{R}_{>0}}\partial_{t,\nu}M(\partial_{t,\nu})g)$ and $u := v + g$.

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Theorem (T.'18)

Define $D_\nu := \{(g(0-), g)\,;\, g \in \mathrm{His}_\nu\}$ *and*

$$T(t) : D_\nu \subseteq \mathrm{IV}_\nu \times \mathrm{His}_\nu \to \mathrm{IV}_\nu \times \mathrm{His}_\nu$$

for $t \geq 0$ *with*

$$T(t)(g(0-), g) := (u(t), \mathbb{1}_{\mathbb{R}_{\leq t}} u).$$

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Note: $T(t)$ not bounded and D_ν not closed!Hille-Yosida condition yields boundedness of $T(t)$ and extension to C_0-sg on $\overline{D_\nu}$.

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Assume $A = 0$ and $M(z) = M_0 + z^{-1}M_1$ for $M_0, M_1 \in L(H)$.
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Theorem (T.'18)

The following are equivalent:

1. $\mathrm{His}_\nu = \{g\,;\ g(0-) \in \mathrm{IV}_\nu\}$,
2. $M(z) = M_0 + z^{-1}M_1$ *for some* $M_0, M_1 \in L(H)$.

Thank you for your attention!

T., *Exponential Stability and Initial Value Problems for Evolutionary Equations.*
Habilitation Thesis, TU Dresden, 2018.