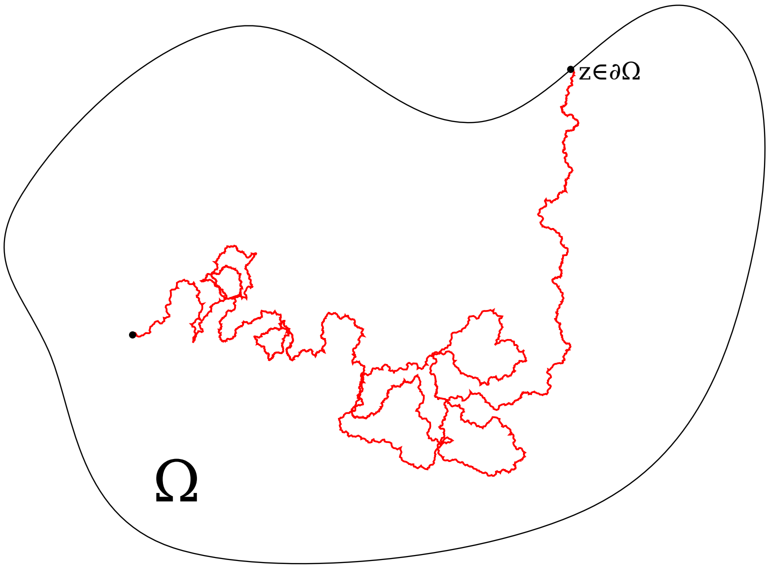
Universität Konstanz

Diffusion with nonlocal boundary conditions on unbounded domains

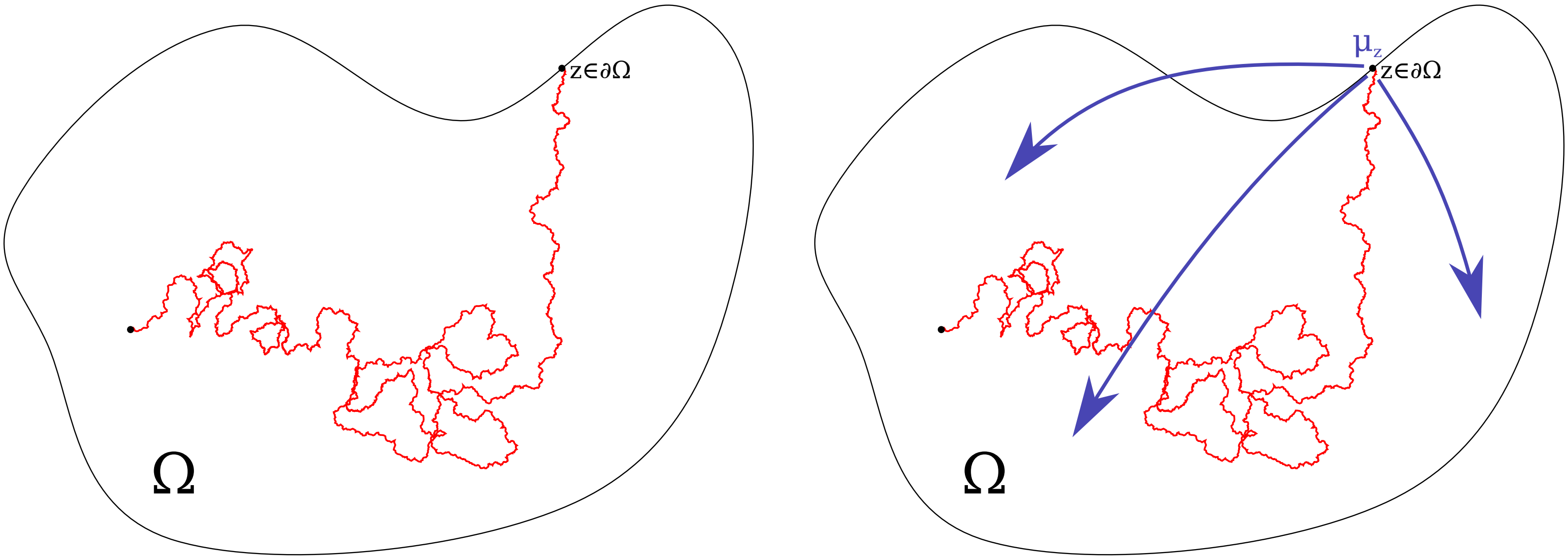
Markus Kunze

Universität Konstanz, October 5th 2018

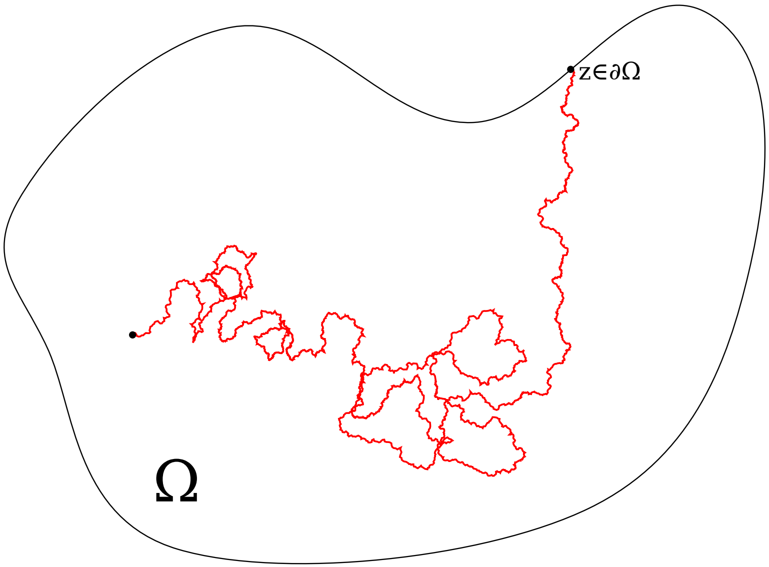
Diffusion

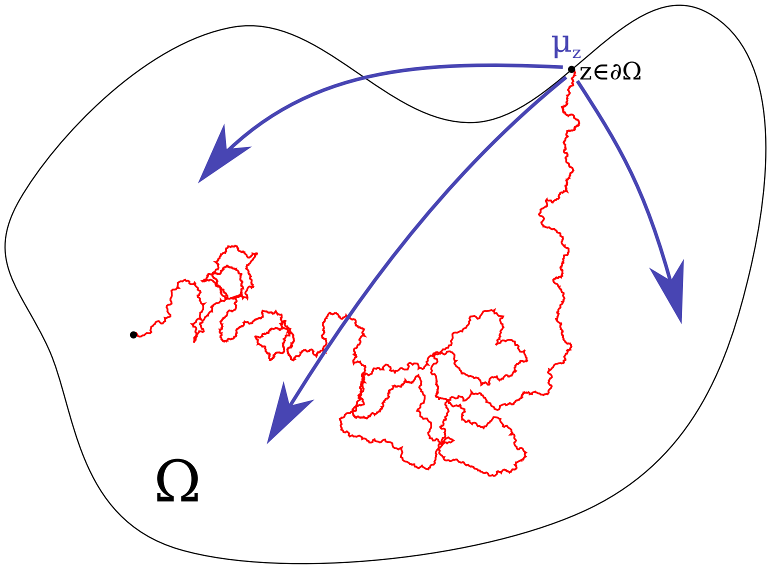


Diffusion with nonlocal boundary conditions



Diffusion with nonlocal boundary conditions





Goal: Study diffusion operator on an unbounded domain with unbounded coefficients subject to nonlocal boundary conditions.

Setting

- $\Omega \subset \mathbb{R}^d$ a (typically unbounded) open and Dirichlet regular set, i.e. in every point $z \in \partial\Omega$ we find a *barrier* at z. This is a function $w \in C(\overline{\Omega \cap B_r(z)})$ with $w(z) = 0$, $w(x) > 0$ for $x \in \Omega \cap B_r(z)$ and $\Delta w \leq 0$ in the distributional sense.

Setting

- $\Omega \subset \mathbb{R}^d$ a (typically unbounded) open and Dirichlet regular set, i.e. in every point $z \in \partial\Omega$ we find a *barrier* at z. This is a function $w \in C(\overline{\Omega \cap B_r(z)})$ with $w(z) = 0$, $w(x) > 0$ for $x \in \Omega \cap B_r(z)$ and $\Delta w \leq 0$ in the distributional sense.
- We are given a map $\mu : \partial\Omega \to \mathscr{M}(\Omega)$ that is
  - $\sigma(\mathscr{M}(\Omega), C_b(\Omega))$-continuous,
  - takes values in the probability measures and
  - there is a ball $B_r(x)$ and $\varepsilon > 0$ such that $\mu(z, B_r(x)) \geq \varepsilon$ for all $z \in \partial\Omega$.

Setting

- $\Omega \subset \mathbb{R}^d$ a (typically unbounded) open and Dirichlet regular set, i.e. in every point $z \in \partial\Omega$ we find a *barrier* at z. This is a function $w \in C(\overline{\Omega \cap B_r(z)})$ with $w(z) = 0$, $w(x) > 0$ for $x \in \Omega \cap B_r(z)$ and $\Delta w \leq 0$ in the distributional sense.
- We are given a map $\mu : \partial\Omega \to \mathscr{M}(\Omega)$ that is
  - $\sigma(\mathscr{M}(\Omega), C_b(\Omega))$-continuous,
  - takes values in the probability measures and
  - there is a ball $B_r(x)$ and $\varepsilon > 0$ such that $\mu(z, B_r(x)) \geq \varepsilon$ for all $z \in \partial\Omega$.
- We set $\mathscr{A} u(x) := \Delta u(x) - \langle x, \nabla u(x)\rangle$ and define the operator A_μ by setting $A_\mu u = \mathscr{A} u$ on

$$D(A_\mu) := \Big\{ u \in C_b(\overline{\Omega}) \cap \bigcap_{1<p<\infty} W^{2,p}_{\text{loc}}(\Omega) : \mathscr{A} u \in L^\infty(\Omega)$$

$$u(z) = \int_\Omega u(x)\,\mu(z, dx) \text{ for all } z \in \partial\Omega \Big\}.$$

Main results

Theorem (K'18) Under the assumptions above:

- The operator A_μ generates a Markovian $*$-semigroup $T_\mu = (T_\mu(t))_{t>0}$ on $L^\infty(\Omega)$.

Main results

Theorem (K'18) Under the assumptions above:

- The operator A_μ generates a Markovian $*$-semigroup $T_\mu = (T_\mu(t))_{t>0}$ on $L^\infty(\Omega)$.
- The semigroup enjoys the strong Feller property, i.e. $T_\mu(t)L^\infty(\Omega) \subset C_b(\overline{\Omega})$ and the restriction to $C_b(\overline{\Omega})$ is given through transition kernels.

Main results

Theorem (K'18) Under the assumptions above:

- The operator A_μ generates a Markovian $*$-semigroup $T_\mu = (T_\mu(t))_{t>0}$ on $L^\infty(\Omega)$.
- The semigroup enjoys the strong Feller property, i.e. $T_\mu(t)L^\infty(\Omega) \subset C_b(\overline{\Omega})$ and the restriction to $C_b(\overline{\Omega})$ is given through transition kernels.
- If Ω is connected, then T_μ has at most one invariant probability measure. If there is an invariant probability measure $\nu^\star$, then

$$T_\mu(t)f \to \int_\Omega f\, d\nu^\star \cdot \mathbb{1}_{\overline{\Omega}} \quad \text{uniformly on compact subsets of } \overline{\Omega}$$

and

$$T_\mu(t)'\nu \to \nu(\overline{\Omega})\nu^\star \quad \text{in total variation norm.}$$

Related results in the Literature

Nonlocal boundary conditions on *bounded* domains.

- *Feller* '52, '54: one-dimensional theory, *immediate return process*.
- *Galakhov, Skubachevskiĭ* '01: Strongly continuous semigroup on $C_\mu(\overline{\Omega})$.
- *Ben-Ari, Pinsky* '07, '09: probabilistic construction of the process.
- *Arendt, Kunkel, K.* '17: Analytic semigroup on $L^\infty(\Omega)$, $C(\overline{\Omega})$.
- ...

Related results in the Literature

Nonlocal boundary conditions on *bounded* domains.

- *Feller* '52, '54: one-dimensional theory, *immediate return process*.
- *Galakhov, Skubachevskiĭ* '01: Strongly continuous semigroup on $C_\mu(\overline{\Omega})$.
- *Ben-Ari, Pinsky* '07, '09: probabilistic construction of the process.
- *Arendt, Kunkel, K.* '17: Analytic semigroup on $L^\infty(\Omega)$, $C(\overline{\Omega})$.
- ...

Diffusion operators with unbounded coefficients

- *Da Prato, Lundardi* '95: Prototype example: Ornstein–Uhlenbeck operator. The semigroup is *not analytic*.
- *Metafune, Pallara, Wacker* '02: General operators on $\mathbb{R}^d$.
- *Fornaro, Metafune, Priola* '04: Dirichlet boundary conditions on unbounded domains.
- *Bertoldi, Fornaro* '04 and *Bertoldi, Fornaro, Lorenzi* '07: Neuman boundary conditions on unbounded domains.
- ...

$*$-semigroups

Let X be the dual of a separable Banach space. We write σ^* for the weak*-topology on X^*.

- A contractive $*$-semigroup is a family $T = (T(t))_{t>0} \subset \mathscr{L}(X^*, \sigma^*)$ such that
  - $T(t+s) = T(t)T(s)$ for all $t, s > 0$,
  - $\|T(t)\| \leq 1$ for all $t > 0$ and
  - for all $x \in X$ and $x^* \in X^*$ the map $t \mapsto \langle T(t)x^*, x\rangle$ is measurable.

$*$-semigroups

Let X be the dual of a separable Banach space. We write σ^* for the weak*-topology on X^*.

- A contractive $*$-semigroup is a family $T = (T(t))_{t>0} \subset \mathscr{L}(X^*, \sigma^*)$ such that
  - $T(t+s) = T(t)T(s)$ for all $t, s > 0$,
  - $\|T(t)\| \leq 1$ for all $t > 0$ and
  - for all $x \in X$ and $x^* \in X^*$ the map $t \mapsto \langle T(t)x^*, x\rangle$ is measurable.
- For $\mathrm{Re}\lambda > 0$, define $R(\lambda) \in \mathscr{L}(X^*, \sigma^*)$ by

$$\langle R(\lambda)x^*, x\rangle = \int_0^\infty e^{-\lambda t}\langle T(t)x^*, x\rangle\, dt.$$

- $(R(\lambda))_{\mathrm{Re}\lambda>0}$ is a pseudoresolvent that determines T uniquely.

$*$-semigroups

Let X be the dual of a separable Banach space. We write σ^* for the weak*-topology on X^*.

- A contractive $*$-semigroup is a family $T = (T(t))_{t>0} \subset \mathscr{L}(X^*, \sigma^*)$ such that
  - $T(t+s) = T(t)T(s)$ for all $t, s > 0$,
  - $\|T(t)\| \leq 1$ for all $t > 0$ and
  - for all $x \in X$ and $x^* \in X^*$ the map $t \mapsto \langle T(t)x^*, x\rangle$ is measurable.
- For $\mathrm{Re}\lambda > 0$, define $R(\lambda) \in \mathscr{L}(X^*, \sigma^*)$ by

$$\langle R(\lambda)x^*, x\rangle = \int_0^\infty e^{-\lambda t}\langle T(t)x^*, x\rangle \, dt.$$

- $(R(\lambda))_{\mathrm{Re}\lambda>0}$ is a pseudoresolvent that determines T uniquely.
- If $\ker R(\lambda) = \{0\}$ for one/all $\mathrm{Re}\lambda > 0$, then there exists an operator A with $R(\lambda, A) = R(\lambda)$. We call A the generator of T.

A monotone convergence theorem for $*$-semigroups

Assume additionally, that X is a KB-space, e.g. $X = L^1(\Omega)$.

A monotone convergence theorem for $*$-semigroups

Assume additionally, that X is a KB-space, e.g. $X = L^1(\Omega)$.

Proposition (K. '18)
Let two contractive $*$-semigroups T_1 and T_2 with Laplace transforms R_1 and R_2 be given and assume that T_1 is positive. Then $T_1(t) \leq T_2(t)$ if and only if $R_1(\lambda) \leq R_2(\lambda)$ for all large enough λ.

A monotone convergence theorem for $*$-semigroups

Assume additionally, that X is a KB-space, e.g. $X = L^1(\Omega)$.

Proposition (K. '18)
Let two contractive $*$-semigroups T_1 and T_2 with Laplace transforms R_1 and R_2 be given and assume that T_1 is positive. Then $T_1(t) \leq T_2(t)$ if and only if $R_1(\lambda) \leq R_2(\lambda)$ for all large enough λ.

Proposition (K. '18)
Let $(T_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive and contractive $*$-semigroups with Laplace transform $(R_n)_{n\in\mathbb{N}}$. Then $T(t) := \sup_{n\in\mathbb{N}} T_n(t)$ defines a positive and contractive $*$-semigroup whose Laplace transform is given by $R(\lambda) = \sup_{n\in\mathbb{N}} R_n(\lambda)$ for all $\lambda > 0$.

Sketch of proof of the main theorem

- Set $\Omega_n := \Omega \cap B_{n+1}(0)$. Pick $\rho_n \in C(\mathbb{R}^d)$ such that $\mathbb{1}_{B_n(0)} \leq \rho_n \leq \mathbb{1}_{B_{n+1}(0)}$ and set

$$\mu_n(z, A) := \begin{cases} \rho_n(z) \int_A \rho_n(x)\mu(z, dx), & z \in \partial\Omega_n \cap \partial\Omega \\ 0, & z \in \partial\Omega_n \setminus \partial\Omega. \end{cases}$$

Sketch of proof of the main theorem

- Set $\Omega_n := \Omega \cap B_{n+1}(0)$. Pick $\rho_n \in C(\mathbb{R}^d)$ such that $\mathbb{1}_{B_n(0)} \leq \rho_n \leq \mathbb{1}_{B_{n+1}(0)}$ and set

$$\mu_n(z, A) := \begin{cases} \rho_n(z) \int_A \rho_n(x)\mu(z, dx), & z \in \partial\Omega_n \cap \partial\Omega \\ 0, & z \in \partial\Omega_n \setminus \partial\Omega. \end{cases}$$

- Put $A_n u(x) = \mathscr{A} u$ for $u \in D(A_n)$, defined by

$$D(A_n) := \Big\{ u \in C_b(\overline{\Omega_n}) \cap \bigcap_{1<p<\infty} W^{2,p}_{\text{loc}}(\Omega_n) : \mathscr{A} u \in L^\infty(\Omega_n)$$

$$u(z) = \int_{\Omega_n} u(x)\mu_n(z, dx) \text{ for all } z \in \partial\Omega_n \Big\}.$$

Sketch of proof of the main theorem

- Set $\Omega_n := \Omega \cap B_{n+1}(0)$. Pick $\rho_n \in C(\mathbb{R}^d)$ such that $\mathbb{1}_{B_n(0)} \leq \rho_n \leq \mathbb{1}_{B_{n+1}(0)}$ and set

$$\mu_n(z, A) := \begin{cases} \rho_n(z) \int_A \rho_n(x)\mu(z, dx), & z \in \partial\Omega_n \cap \partial\Omega \\ 0, & z \in \partial\Omega_n \setminus \partial\Omega. \end{cases}$$

- Put $A_n u(x) = \mathscr{A} u$ for $u \in D(A_n)$, defined by

$$D(A_n) := \Big\{ u \in C_b(\overline{\Omega_n}) \cap \bigcap_{1<p<\infty} W^{2,p}_{\mathrm{loc}}(\Omega_n) : \mathscr{A} u \in L^\infty(\Omega_n)$$
$$u(z) = \int_{\Omega_n} u(x)\mu_n(z, dx) \text{ for all } z \in \partial\Omega_n \Big\}.$$

- By results on bounded domains: A_n is the generator of an analytic semigroup T_n on $L^\infty(\Omega_n)$, that is positive and contractive and enjoys the strong Feller property.

Sketch of proof cont'd

- we can view $R(\lambda, A_n)$ and $T_n(t)$ as operators on $L^\infty(\Omega)$, extending functions with zero outside Ω_n ($\rightsquigarrow$ operators take values in $C(\overline{\Omega})$).

Sketch of proof cont'd

- we can view $R(\lambda, A_n)$ and $T_n(t)$ as operators on $L^\infty(\Omega)$, extending functions with zero outside Ω_n ($\rightsquigarrow$ operators take values in $C(\overline{\Omega})$).
- Maximum principle $\rightsquigarrow$ for every $\lambda > 0$ the sequence $R(\lambda, A_n)$ is increasing. Show that $R(\lambda) := \sup_n R(\lambda, A_n)$ is an injective, positive, adjoint operator that takes values in $D(A_\mu)$ and $u := R(\lambda)f$ solves the elliptic equation $\lambda u - \mathscr{A} u = f$.

Sketch of proof cont'd

- we can view $R(\lambda, A_n)$ and $T_n(t)$ as operators on $L^\infty(\Omega)$, extending functions with zero outside Ω_n ($\rightsquigarrow$ operators take values in $C(\overline{\Omega})$).
- Maximum principle $\rightsquigarrow$ for every $\lambda > 0$ the sequence $R(\lambda, A_n)$ is increasing. Show that $R(\lambda) := \sup_n R(\lambda, A_n)$ is an injective, positive, adjoint operator that takes values in $D(A_\mu)$ and $u := R(\lambda)f$ solves the elliptic equation $\lambda u - \mathscr{A} u = f$.
- Use Lyapunov function $V(x) = |x|^2$ and concentration assumption to prove that $\lambda - \mathscr{A}$ is injective on $D(A_\mu)$, from which it follows that $R(\lambda) = R(\lambda, A_\mu)$.

Sketch of proof cont'd

- we can view $R(\lambda, A_n)$ and $T_n(t)$ as operators on $L^\infty(\Omega)$, extending functions with zero outside Ω_n ($\rightsquigarrow$ operators take values in $C(\overline{\Omega})$).
- Maximum principle $\rightsquigarrow$ for every $\lambda > 0$ the sequence $R(\lambda, A_n)$ is increasing. Show that $R(\lambda) := \sup_n R(\lambda, A_n)$ is an injective, positive, adjoint operator that takes values in $D(A_\mu)$ and $u := R(\lambda)f$ solves the elliptic equation $\lambda u - \mathscr{A} u = f$.
- Use Lyapunov function $V(x) = |x|^2$ and concentration assumption to prove that $\lambda - \mathscr{A}$ is injective on $D(A_\mu)$, from which it follows that $R(\lambda) = R(\lambda, A_\mu)$.
- By the monotone convergence theorem, A_μ is the generator of a positive and contractive $*$-semigroup T_μ.

Sketch of proof cont'd

- we can view $R(\lambda, A_n)$ and $T_n(t)$ as operators on $L^\infty(\Omega)$, extending functions with zero outside Ω_n ($\rightsquigarrow$ operators take values in $C(\overline{\Omega})$).
- Maximum principle $\rightsquigarrow$ for every $\lambda > 0$ the sequence $R(\lambda, A_n)$ is increasing. Show that $R(\lambda) := \sup_n R(\lambda, A_n)$ is an injective, positive, adjoint operator that takes values in $D(A_\mu)$ and $u := R(\lambda)f$ solves the elliptic equation $\lambda u - \mathscr{A} u = f$.
- Use Lyapunov function $V(x) = |x|^2$ and concentration assumption to prove that $\lambda - \mathscr{A}$ is injective on $D(A_\mu)$, from which it follows that $R(\lambda) = R(\lambda, A_\mu)$.
- By the monotone convergence theorem, A_μ is the generator of a positive and contractive $*$-semigroup T_μ.
- Strong Feller property: $0 \leq f \leq 1$ $\rightsquigarrow T_\mu(t)f = \sup_n T_n f$ is lower semicontinuous as supremum of continuous functions.

Sketch of proof cont'd

- we can view $R(\lambda, A_n)$ and $T_n(t)$ as operators on $L^\infty(\Omega)$, extending functions with zero outside Ω_n ($\rightsquigarrow$ operators take values in $C(\overline{\Omega})$).
- Maximum principle $\rightsquigarrow$ for every $\lambda > 0$ the sequence $R(\lambda, A_n)$ is increasing. Show that $R(\lambda) := \sup_n R(\lambda, A_n)$ is an injective, positive, adjoint operator that takes values in $D(A_\mu)$ and $u := R(\lambda)f$ solves the elliptic equation $\lambda u - \mathscr{A} u = f$.
- Use Lyapunov function $V(x) = |x|^2$ and concentration assumption to prove that $\lambda - \mathscr{A}$ is injective on $D(A_\mu)$, from which it follows that $R(\lambda) = R(\lambda, A_\mu)$.
- By the monotone convergence theorem, A_μ is the generator of a positive and contractive $*$-semigroup T_μ.
- Strong Feller property: $0 \leq f \leq 1$ $\rightsquigarrow T_\mu(t)f = \sup_n T_n f$ is lower semicontinuous as supremum of continuous functions.
- On the other hand, $\mathbb{1} \in \ker A_\mu$ so $T_\mu(t)\mathbb{1} \equiv \mathbb{1}$. Thus $\mathbb{1} - T_\mu(t)f = \sup_n T_n(t)(\mathbb{1} - f)$ is also lower semicontinuous, whence $T_\mu f$ is upper semicontinuous. Altogether, $T_\mu(t)f$ is continuous.

Aysmptotic behavior

- Using again the Lyapunov function $V(x) = |x|^2$ one can show that if Ω is connected, then $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$.

Aysmptotic behavior

- Using again the Lyapunov function $V(x) = |x|^2$ one can show that if Ω is connected, then $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$.
- As T_μ enjoys the strong Feller property, it follows from results of *Gerlach, K.* '14 that T_μ has at most one invariant probability measure.

Aysmptotic behavior

- Using again the Lyapunov function $V(x) = |x|^2$ one can show that if Ω is connected, then $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$.
- As T_μ enjoys the strong Feller property, it follows from results of *Gerlach, K.* '14 that T_μ has at most one invariant probability measure.
- If T_μ has an invariant probability measure, the claimed asymptotic behavior follows from recent results of *Gerlach, Glück* '18+.

Aysmptotic behavior

- Using again the Lyapunov function $V(x) = |x|^2$ one can show that if Ω is connected, then $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$.
- As T_μ enjoys the strong Feller property, it follows from results of *Gerlach, K.* '14 that T_μ has at most one invariant probability measure.
- If T_μ has an invariant probability measure, the claimed asymptotic behavior follows from recent results of *Gerlach, Glück* '18+.

Theorem (K. '18)

Assume that $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$ and that there exists a function $V \in C(\overline{\Omega}) \cap \bigcap_{1<p<\infty} W^{2,p}_{\mathrm{loc}}(\Omega)$ such that

1. $V \geq 0$ and $V(x) \to \infty$ as $|x| \to \infty$;
2. $\mathscr{A}V$ has a version that is continuous on Ω, bounded on bounded subsets of Ω and satisfies $\mathscr{A}V(x) \to -\infty$ as $|x| \to \infty$.
3. V is integrable with respect to $\mu(z)$ and $\int V(x)\mu(z,dx) \leq V(z)$ for all $z \in \partial\Omega$.

Then $T\mu$ has a unique invariant probability measure.

An example

- Consider $d = 1$, $\Omega = (0, \infty)$ and $\mathscr{A}u(x) = u''(x) - xu'(x)$.

An example

- Consider $d = 1$, $\Omega = (0, \infty)$ and $\mathscr{A} u(x) = u''(x) - xu'(x)$.
- $V(x) = x^2$ satisfies points 1. and 2., as $\mathscr{A} u(x) = 2 - 2x^2$.

An example

- Consider $d = 1$, $\Omega = (0, \infty)$ and $\mathscr{A} u(x) = u''(x) - xu'(x)$.
- $V(x) = x^2$ satisfies points 1. and 2., as $\mathscr{A} u(x) = 2 - 2x^2$.
- Suppose V is integrable with respect to $\mu = \mu(0)$. Pick $0 < r < R$ such that

$$\mu((0, r)) \leq \frac{1}{2} \quad \text{and} \quad \int_R^\infty V(x)\, d\mu(x) \leq \frac{1}{2}.$$

An example

- Consider $d = 1$, $\Omega = (0, \infty)$ and $\mathscr{A} u(x) = u''(x) - xu'(x)$.
- $V(x) = x^2$ satisfies points 1. and 2., as $\mathscr{A} u(x) = 2 - 2x^2$.
- Suppose V is integrable with respect to $\mu = \mu(0)$. Pick $0 < r < R$ such that

$$\mu((0,r)) \leq \frac{1}{2} \quad \text{and} \quad \int_R^\infty V(x)\, d\mu(x) \leq \frac{1}{2}.$$

- We find $0 \leq \tilde{V} \in C^2(0,\infty) \cap C([0,\infty)$ with $\tilde{V}(0) = 1$, $\tilde{V}(x) = 0$ for $x \in [r, R]$ and $\tilde{V}(x) = x^2$ for $x \geq R + 1$. We may arrange $\tilde{V}(x) \leq 1$ for $x \in [0, r]$ and $\tilde{V}(x) \leq V(x)$ for $x \in [R, R+1]$. Then $\tilde{V}$ satisfies 1., 2. and 3.

An example

- Consider $d = 1$, $\Omega = (0, \infty)$ and $\mathscr{A} u(x) = u''(x) - xu'(x)$.
- $V(x) = x^2$ satisfies points 1. and 2., as $\mathscr{A} u(x) = 2 - 2x^2$.
- Suppose V is integrable with respect to $\mu = \mu(0)$. Pick $0 < r < R$ such that

$$\mu((0, r)) \leq \frac{1}{2} \quad \text{and} \quad \int_R^\infty V(x)\, d\mu(x) \leq \frac{1}{2}.$$

- We find $0 \leq \tilde{V} \in C^2(0, \infty) \cap C([0, \infty)$ with $\tilde{V}(0) = 1$, $\tilde{V}(x) = 0$ for $x \in [r, R]$ and $\tilde{V}(x) = x^2$ for $x \geq R + 1$. We may arrange $\tilde{V}(x) \leq 1$ for $x \in [0, r]$ and $\tilde{V}(x) \leq V(x)$ for $x \in [R, R + 1]$. Then $\tilde{V}$ satisfies 1., 2. and 3.
- We may argue similar on exterior domains.

The end

Thank you for your attention

Referenzen:

- W. Arendt, S. Kunkel, M.K., *Diffusion with nonlocal boundary conditions*, J. Funct. Anal., Vol. 270(7), 2017, 663–706.
- W. Arendt, S. Kunkel, M.K., *Diffusion with nonlocal Robin boundary conditions*, J. Math. Soc. Japan, Vol. 70(4), 2018, 1523–1556
- M.K., *Diffusion with nonlocal Dirichlet boundary conditions on unbounded domains*, in preparation.