Heat kernel of isotropic nonlocal operators

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Semigroups of Operators: Theory and Applications
Kazimierz Dolny, October 4, 2018

The talk is based on 3 papers (2014-2015):

Density and tails of unimodal convolution semigroups, JFA
Barriers, exit time and survival pr. for unimodal Lévy processes, PTRF
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- Consider $a=0$ and $L\varphi(x) = PV \int_{\mathbb{R}^d} (\varphi(x+y)-\varphi(x))\nu(y)dy$.

Examples

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- We will assume $p_t(0) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-t\psi(\xi)}\mathrm{d}\xi < \infty$.

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$$\mathbb{P}_x(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n) = \int_{B_1} p_{t_1}(x_1 - x) \cdots \int_{B_n} p_{t_n - t_{n-1}}(x_n - x_{n-1})\, dx_n \cdots dx_1.$$

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$$G_D(x, y) := \int_0^\infty p_t^D(x, y) \mathrm{d}t.$$

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$$p_t^D(x,y) := p_t(y-x) - \mathbb{E}_x\left[\tau_D \le t;\ p_{t-\tau_D}(y - X_{\tau_D})\right]$$

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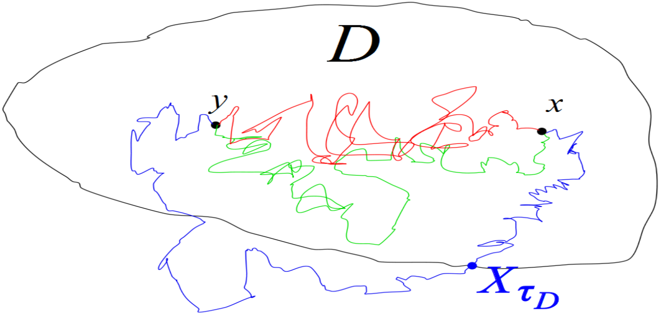

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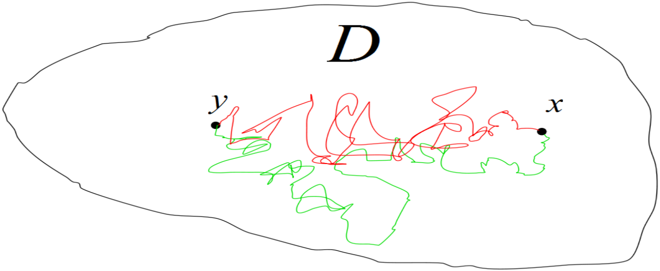

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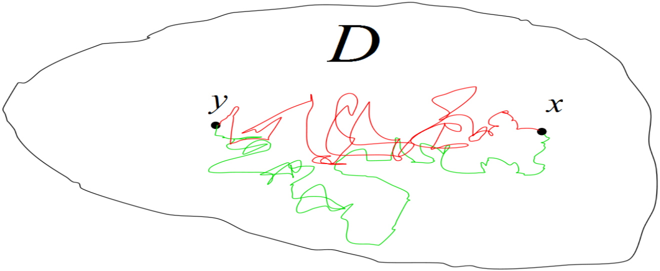
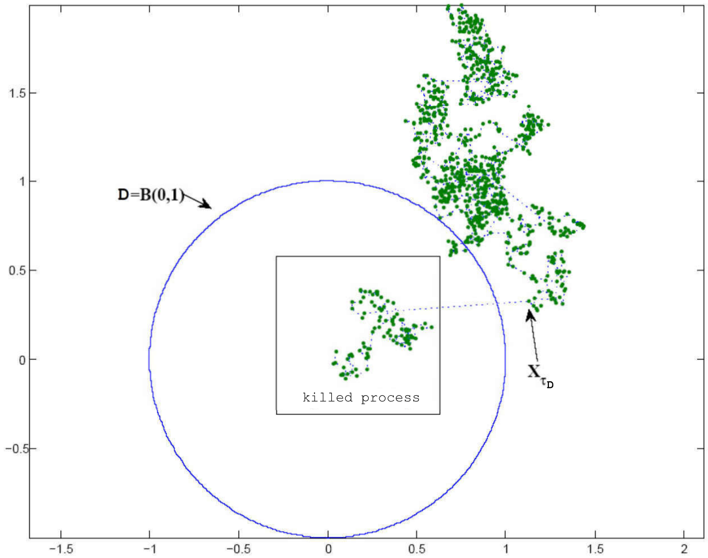


E.g., probability for X to survive time t is $\mathbb{P}_x(\tau_D > t) = \int p_t^D(x,y)dy$.

Simulated trajectory $t \mapsto X_t$, for $\alpha = 1.8$, $d = 2$



Ikeda-Watanabe formula

If $x \in D$, then the $\mathbb{P}_x$ joint density function of $(\tau_D, X_{\tau_D -}, X_{\tau_D})$ is:

$$(0, \infty) \times D \times \left(\overline{D}\right)^c \ni (s, u, z) \mapsto p_s^D(x, u)\nu(z - u).$$

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$$\mathbb{P}_x(X_{\tau_D} \in A) = \int_D G_D(x,u)\nu(A-u)du \quad \text{if } \operatorname{dist}(A,D) > 0.$$

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Example: relativistic α-stable process, $0 < \alpha < 2$:

$\psi(\xi) = (|\xi|^2 + 1)^{\alpha/2} - 1 \approx |\xi|^2 \wedge |\xi|^\alpha$ has LS$(\alpha, 0)$, US$(\alpha, 1)$.

(subcritical) Scaling conditions

Let $\psi^*(u) = \sup_{0 \le r \le u} \psi(r)$. We have $\psi(u) \le \psi^*(u) \le \pi^2 \psi(u)$, $u \ge 0$.

Thus $\psi \approx \psi^*$, and we call ψ **almost increasing**.

Definition: $\psi \in$LS$(\underline{\alpha}, \underline{\theta})$ **if** $(\underline{\theta}, \infty) \ni u \mapsto \psi(u)/u^{\underline{\alpha}}$ almost increases.

Definition: $\psi \in$US$(\overline{\alpha}, \overline{\theta})$ **if** $(\overline{\theta}, \infty) \ni u \mapsto \psi(u)/u^{\overline{\alpha}}$ almost decreases.

We always assume here that $0 < \underline{\alpha} \le \overline{\alpha} < 2$ (subcritical).

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Example: $\psi(\xi) = |\xi|^\beta + |\xi|^\alpha \approx |\xi|^\beta \vee |\xi|^\alpha$, where $0 < \beta < \alpha < 2$.

“Common bounds” for isotropic semigroups with scalings

Theorem (Bogdan, Grzywny, Ryznar (2014))

If (subcritical) LS *and* US *hold, then locally (or globally) in space and time,*

$$p_t(x) \approx \psi^{-1}(1/t)^d \ \wedge \ \frac{t\psi(1/|x|)}{|x|^d} \approx p_t(0) \wedge \ t\nu(x).$$

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Example: $X_t = X_t^{(\alpha)} + X_t^{(\beta)}, \quad 0 < \beta < \alpha < 2$ (Chen, Kumagai 2008);

$$p_t(x) \approx \min\left\{t^{-d/\alpha} \wedge t^{-d/\beta},\ t\left(\frac{1}{|x|^{d+\alpha}} + \frac{1}{|x|^{d+\beta}}\right)\right\}, \quad t > 0,\ x \in \mathbb{R}^d.$$

Theorem (Bogdan, Grzywny, Ryznar (2014))

For unimodal probability convolution semigroup $\{p_t\}$ on $\mathbb{R}^d$, with Lévy-Khintchine exponent ψ and Lévy measure density ν, TFAE:

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Observe T. Grzywny [2016-] and A. Mimica [2018] for critical scalings...

The Dirichlet heat kernel

$$p_t^D(x, y)$$

The renewal function V

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Corollary: If $\psi \in \mathrm{LS}(\underline{\alpha}, 0) \cap \mathrm{US}(\overline{\alpha}, 0)$ (global subcritical scalings), then

$$p_t(x) \approx V^{-1}(\sqrt{t})^{-d} \;\wedge\; \frac{t}{V^2(|x|)|x|^d}, \qquad t > 0,\, x \in \mathbb{R}^d.$$

In proofs we often use the Pruitt's function

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This follows by Dynkin's formula:

$$\mathbb{E}_x g(X_{\tau_D}) = g(x) + \mathbb{E}_x \int_0^{\tau_D} Lg(X_s)ds, \quad x \in \mathbb{R}^d.$$

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$$4 \sup_{u \geq 0} |\phi'(u)| + \frac{1}{2} \sup_{u \geq 0} |\phi''(u)| \;=\; 24.$$

Dynkin approximate operator (orientation: $\mathcal{A}_\epsilon \to \mathcal{A} \supset L$)

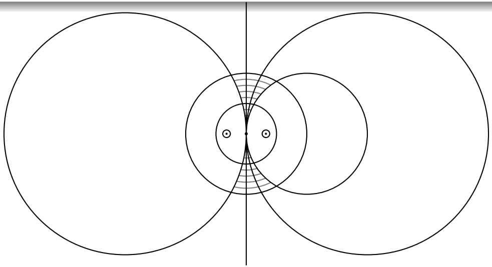
Let $\mathcal{A}_\epsilon g(x) = \mathbb{E}_x[g(X_{\tau_{B(x,\epsilon)}}) - g(x)]/\mathbb{E}_x\tau_{B(x,\epsilon)}$, where $\epsilon > 0$.

We assume a certain Harnack-type condition (**H**) on V'.

Theorem

Let $x_0 \in \mathbb{R}^d$, $r > 0$ and $g(x) = V(\delta_{B(x_0,r)}(x))$. Then,

$$0 \leq \limsup_{\epsilon\to 0} \left[-\mathcal{A}_\epsilon g(x) \right] \leq \frac{C}{V(r)} \qquad \textit{if} \quad 0 < \delta_{B(x_0,r)}(x) < r/4.$$



Applications of superharmonic functions (barriers)

Corollary

If $(\mathbf{H})$ *holds, then* $s_r(x) := \mathbb{E}_x \tau_{B_r} \approx V(\delta_{B_r}(x))V(r), \quad x \in \mathbb{R}^d.$

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Lipschitz domains require different tools...

Bounded domains: some spectral theory

If D is an open bounded set and $p_t(0)$ is bounded for every $t > 0$, then the integral operators on $L^2(D)$ with kernels $p_t^D(x, y) \leq p_t(0)$ are compact.

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The eigenvalues $0 < \lambda_1 < \lambda_2 \leq \dots$ and the orthonormal basis of eigenfunctions $\phi_1 \geq 0, \phi_2, \phi_3 \dots$ satisfy

$$\phi_k(x) = e^{\lambda_k t} \int p_t^D(x, z)\phi_k(z)dz.$$

Bounded smooth D

Theorem

If ψ has LS and US and $D \subset \mathbb{R}^d$ is a bounded $C^{1,1}$ open set, then

$$p_t^D(x,y) \approx \mathbb{P}_x(\tau_D > t/2)\; p_{t\wedge t_0}(x-y)\; \mathbb{P}_y(\tau_D > t/2),$$

for all $t > 0$, $x, y \in D$, and we have

$$\mathbb{P}_x(\tau_D > t) \approx e^{-\lambda_1 t} \left(\frac{V(\delta_D(x))}{\sqrt{t \wedge t_0}} \wedge 1 \right),$$

where $t_0 = V^2(r_0)$, and $r_0 > 0$ is sufficiently small.

Exterior domains

Theorem (Bogdan, Grzywny, Ryznar (2014))

Let $\psi \in \mathrm{LS}(\underline{\alpha}, 0) \cap \mathrm{US}(\overline{\alpha}, 0)$ *and* $d > \overline{\alpha}$. *Let* D *be a* $C^{1,1}$ *at scale* r *and* $D^c \subset \overline{B_R}$. *For all* $x, y \in \mathbb{R}^d$ *and* $t > 0$ *we have*

$$p_t^D(x,y) \approx \mathbb{P}_x(\tau_D > t)\; p_t(x-y)\; \mathbb{P}_y(\tau_D > t),$$

and

$$\mathbb{P}_x(\tau_D > t) \approx \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(r)} \wedge 1$$

with comparability constants $C = C(d, \psi, R/r)$.

Exterior domains: examples

Example: Let $\psi(\xi) = |\xi|^\alpha$, $\alpha \in (0,2)$, and $D = \overline{B(0,1)}^c$. Then,

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Halfspace-like domains

Let ψ satisfy global LS and US, $\mathbb{H}_a := \{(x_1, \ldots, x_d) : x_d > a\}$.

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Let D be $C^{1,1}$ at scale R and $\mathbb{H}_a \subset D \subset \mathbb{H}_b$. For all $x, y \in \mathbb{R}^d$ and $t > 0$,

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- Isotropic stable processes
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- See [R. Schilling, R. Song and Z. Vondraček. 2012] for much more

Two samples for flavor to take out:

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Lemma

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Proposition (Bogdan, Grzywny, Ryznar (ruin probability))

For every symmetric Lévy process in $\mathbb{R}$ which is not compound Poisson,

$$\mathbb{P}_x(\tau_{(0,\infty)} \geq t) \approx 1 \wedge \frac{V(x)}{\sqrt{t}}, \quad t > 0,\ x \in \mathbb{R}.$$

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On upper bound of the Dirichlet kernel

Lemma

Consider open $D_1, D_3 \subset D$ *such that* $\mathrm{dist}(D_1, D_3) > 0$*. Let* $D_2 = D \setminus (D_1 \cup D_3)$*. If* $x \in D_1$*,* $y \in D_3$ *and* $t > 0$*, then*

$$\begin{aligned} p_t^D(x,y) &\le \mathbb{P}_x(X_{\tau_{D_1}} \in D_2) \sup_{s<t,\, z\in D_2} p(s,z,y) \\ &+ (t \wedge \mathbb{E}_x \tau_{D_1}) \sup_{u\in D_1,\, z\in D_3} \nu(z-u)\,. \end{aligned}$$

On lower bound of the Dirichlet kernel

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$$p_t^D(x,y) \geq t\mathbb{P}_x(\tau_{D_1} > t)\,\mathbb{P}_y(\tau_{D_3} > t) \inf_{u \in D_1,\, z \in D_3} \nu(z-u)\,.$$