Hot spots of quantum graphs

James Kennedy

Group of Mathematical Physics
University of Lisbon

Ongoing joint work with Jonathan Rohleder (Stockholm)

Semigroups of Operators: Theory and Applications
Kazimierz Dolny

Tuesday, 2 October, 2018

$\langle \phi | \mu \rangle$ Grupo de Física Matemática da Universidade de Lisboa

FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR

The Hot Spots Conjecture

Conjecture (J. Rauch, 1974)

The hottest and coldest points within a perfectly insulated body should converge to the boundary of the body for large times.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary and consider the Neumann Laplacian

$$D(\Delta_\Omega^N) = \left\{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega),\ \frac{\partial u}{\partial \nu} = 0 \text{ in } L^2(\partial\Omega) \right\}$$
$$\Delta_\Omega^N u = \Delta u$$

For an initial condition u_0 the diffusion of heat in Ω described by

$$u(t,x) = e^{t\Delta_\Omega^N} u_0(x), \qquad x \in \Omega,\ t > 0.$$

The Hot Spots Conjecture

Let $0 = \mu_1 \leq \mu_2 \leq \ldots$ be the eigenvalues and $\psi_1, \psi_2, \ldots$ the normalised eigenfunctions of $-\Delta_\Omega^N$, then by the spectral theorem

$$u = e^{t\Delta_\Omega^N} u_0 = \sum_{k=1}^{\infty} \langle u_0, \psi_k \rangle_{L^2(\Omega)} e^{-t\mu_k} \psi_k.$$

Since ψ_1 is constant, for a "generic" initial condition u_0 the second eigenfunction(s) ψ_2 determine(s) the profile of u for $t \to \infty$. Hence the most common formulation of the conjecture is:

The Hot Spots Conjecture

Let $\Omega \subset \mathbb{R}^d$ be a bounded, smooth domain and ψ_2 any eigenfunction associated with the second Neumann eigenvalue μ_2. Then

$$\max_{x \in \overline{\Omega}} \psi_2(x), \qquad \min_{x \in \overline{\Omega}} \psi_2(x)$$

are achieved (only) on $\partial\Omega$.

The Hot Spots Conjecture

- ...is true for intervals! If $\Omega = (0, 1)$ then $\psi_2(x) = \cos(\pi x)$ with maximum at 0 and minimum at 1. Similar story for balls, rectangles, "long thin domains", ...
- ...is not true for all domains in $\mathbb{R}^d$. (Burdzy and Werner, Ann. of Math., 1999)
- ...is (probably) true for triangles. (Polymath project of Tao; works of Bañuelos, Burdzy, Siudeja; preprint (2018) of Judge and Mondal)
- ...is open for general *convex* domains, even in two dimensions.

Quantum Graphs

Now suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a connected, compact metric graph:

- $\mathcal{V}$ is a finite vertex set,
- $\mathcal{E}$ is a finite edge set, and each edge can be identified with an interval of finite length,
- multiple parallel edges (i.e., more than one edge running between the same two vertices) and loops are allowed,

and consider the Laplacian $-\Delta$ on $\mathcal{G}$ with standard vertex conditions:

- $-\Delta = -\frac{d^2}{dx^2}$ on each edge,
- Functions in the domain of $-\Delta$ are continuous on $\mathcal{G}$ and satisfy the Kirchhoff condition at each vertex (flow in equals flow out),
- Kirchhoff equals Neumann in a vertex of degree one,
- The operator Δ generates a C_0-semigroup which determines diffusion on a "perfectly insulated" graph,
- $\mu_1 = 0$ with eigenfunction constant,
- $\mu_2 > 0$ and its eigenfunction(s) ψ_2 change sign in $\mathcal{G}$.

Hot Spots of Quantum Graphs

Question

Where are the maximum and minimum of ψ_2 located, and how does this relate to the geometry of $\mathcal{G}$?

Some definitions:

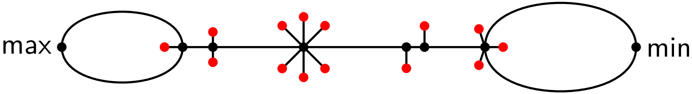
- $M := \{x \in \mathcal{G} : \exists\, \psi_2$ achieving its global maximum on $\mathcal{G}$ at $x\}$,
- $M_{loc} := \{x \in \mathcal{G} : \exists\, \psi_2$ achieving a (nonzero) local maximum on $\mathcal{G}$ at $x\} \supset M$,
- $\partial\mathcal{G} := \{v \in \mathcal{V} : \deg v = 1\}$.

Some (naïve) questions:

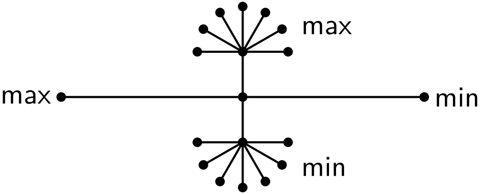
- Do we have a "hot spots theorem" for quantum graphs: $M \subset \partial\mathcal{G}$? If so, this would suggest that $\partial\mathcal{G}$ is an (analytically) "good" notion of boundary
- Does M realise the diameter of $\mathcal{G}$, i.e., can one find $x, y \in M$ s.t. $\operatorname{dist}(x, y) = \operatorname{diam}\mathcal{G}$? (Or at least $\operatorname{dist}(x, y) \cong \operatorname{diam}\mathcal{G}$?)

Some (counter-) examples

M need not have anything to do with $\partial\mathcal{G}$:



M need not have anything to do with $\mathrm{diam}\,\mathcal{G}$:



Hot Spots of *Tree Graphs*

Suppose $\mathcal{T}$ is a (compact) tree, i.e., $\mathcal{T}$ has no cycles, and recall M is the set of global maxima (and minima), M_{loc} is the set of local maxima (and minima). Then:

Theorem (K.–Rohleder, 2018)

(1) $M \subset M_{loc} \subset \partial\mathcal{T}$;

(2) if ψ_2 does not vanish identically on any edge, then $M_{loc} = \partial\mathcal{T}$;

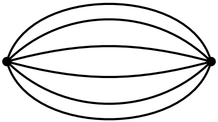
(3) $\#M = 2$ *generically*

("Generically": consider all possible edge lengths for a given graph topology. A property holds *generically* if the set of edge lengths for which it holds is of the second Baire category in $\mathbb{R}_+^{\#\mathcal{E}}$.)

What can we say about M (and M_{loc}) in general?

Return to considering a general (connected, compact) graph $\mathcal{G}$.

- $M = \mathcal{G}$ is possible (loops, equilateral pumpkin graphs, equilateral complete graphs)



- Conjecture: either $M = \mathcal{G}$ or M is finite, and the same is true of M_{loc}
- Observation: M and M_{loc} are finite whenever μ_2 is simple, and μ_2 is simple generically. In particular, $M = \mathcal{G}$ is "rare"
- Observation: $\partial\mathcal{G} \subset M_{loc}$ if ψ_2 does not vanish identically on any edge (and generically it doesn't)
- Conjecture: generically, $\#M = 2$. Thus for most graphs there are two "distinguished" points where the heat (or cold) is asymptotically most concentrated

Thank you for your attention!