

Convolution semigroups on quantum groups and non-commutative Dirichlet forms

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Semigroups of Operators: Theory and Applications
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A C^* -algebra is...

an involutive Banach algebra such that $\|x^*x\| = \|x\|^2$ for all x .

- Commutative case: $C_0(\Omega)$

A von Neumann algebra is...

a $*$ -subalgebra of $B(\mathcal{H})$, containing I and SOT-closed.

- Commutative case: $L^\infty(X, \mathbb{A}, \mu)$ over $\mathcal{H} = L^2(X, \mathbb{A}, \mu)$

A **state** of a C^* -algebra \mathcal{A} is a functional on \mathcal{A} that maps positive elements into $[0, \infty)$ and has norm 1.

Example

States of $C_0(\Omega) \longleftrightarrow$ regular probability measures on Ω .

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A group as a quantum group

G – locally compact group.

- 1 $L^\infty(G)$ – a von Neumann algebra.
- 2 Co-multiplication: the $*$ -homomorphism

$$\Delta : L^\infty(G) \rightarrow L^\infty(G) \bar{\otimes} L^\infty(G) \cong L^\infty(G \times G)$$

defined by

$$(\Delta(f))(t, s) := f(ts) \quad (f \in L^\infty(G)).$$

By associativity, we have $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$.

- 3 Left and right Haar measures. View them as functions $\varphi, \psi : L^\infty(G)_+ \rightarrow [0, \infty]$ by $\varphi(f) := \int_G f(t) dt_\ell$, $\psi(f) := \int_G f(t) dt_r$.

Motivation for quantum groups

Lack of Pontryagin duality for non-abelian l.c. groups.

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Definition (Kustermans–Vaes, '00)

A **locally compact quantum group** is a pair $\mathbb{G} = (M, \Delta)$ such that:

- 1 M is a **von Neumann algebra**
- 2 $\Delta : M \rightarrow M \bar{\otimes} M$ is a **co-multiplication**: a normal, faithful, unital $*$ -homomorphism which is co-associative, i.e.,

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$$

- 3 There are two n.s.f. weights φ, ψ on M (the **Haar weights**) with:
 - ▶ $\varphi((\omega \otimes \text{id})\Delta(x)) = \omega(\mathbb{1})\varphi(x)$ when $\omega \in M_*^+$, $x \in M^+$ and $\varphi(x) < \infty$
 - ▶ $\psi((\text{id} \otimes \omega)\Delta(x)) = \omega(\mathbb{1})\psi(x)$ when $\omega \in M_*^+$, $x \in M^+$ and $\psi(x) < \infty$.

Denote $L^\infty(\mathbb{G}) := M$.

Have it act standardly on the Hilbert space $L^2(\mathbb{G})$.

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Locally compact quantum groups

Features

Duality $G \mapsto \hat{G}$ within the category satisfying $\hat{\hat{G}} = G$.

Three “faces” (algebras): the von Neumann algebra $L^\infty(G)$, the “reduced” C^* -algebra $C_0(G)$, and the “universal” C^* -algebra $C_0^u(G)$.

The **antipode**: an (unbounded) operator on $L^\infty(G) / C_0(G) / C_0^u(G)$.
Decomposes into:

- a “bounded part”: the **unitary antipode**, an anti-automorphism;
- an “unbounded part”: the **scaling group**.

Two basic examples

- $G = G$
- $G = \hat{G}$

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Convolution semigroups

\mathbb{G} – locally compact quantum group.

The conjugate space $C_0^u(\mathbb{G})^*$ carries a convolution \star turning it into a Banach algebra with unit ϵ (the co-unit).

Definition

A **convolution semigroup of states on \mathbb{G}** is a family $(\mu_t)_{t \geq 0}$ of states of $C_0^u(\mathbb{G})$ such that

$$\mu_0 = \epsilon \quad \text{and} \quad \mu_s \star \mu_t = \mu_{s+t} \quad (\forall s, t \geq 0).$$

Adjectives:

- w^* -continuous
- **symmetric** = invariant under the unitary antipode.

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Examples

of w^* -continuous, symmetric, convolution semigroups of states on G :

- 1 $G = G$: w^* -continuous, symmetric, convolution semigroups of probability measures on G ;

that is:

families $(\mu_t)_{t \geq 0}$ of probability Borel measures on G satisfying

$$\mu_0 = \delta_e \quad \text{and} \quad \mu_s \star \mu_t = \mu_{s+t} \quad (\forall s, t \geq 0)$$

that are w^* -continuous and invariant under “inversion of sets”.

Examples

of w^* -continuous, symmetric, convolution semigroups of states on G :

- $G = \hat{G}$: w^* -continuous, symmetric, semigroups of normalized positive-definite functions on G ;

that is:

families $(\varphi_t)_{t \geq 0}$ of normalized positive-definite functions on G satisfying

$$\varphi_0 \equiv 1 \quad \text{and} \quad \varphi_s \cdot \varphi_t = \varphi_{s+t} \quad (\forall s, t \geq 0)$$

that are w^* -continuous and invariant under inversion (= real).

Dirichlet forms and Markov semigroups

Definition

A (non-negative) **quadratic form** on a Hilbert space \mathcal{H} is a semi-inner product $Q : D(Q) \times D(Q) \rightarrow \mathbb{C}$ on a subspace $D(Q)$ of \mathcal{H} .

- **Densely defined** if $D(Q)$ is dense in \mathcal{H} .
- **Closedness**.

More convenient to work with $Q' : \mathcal{H} \rightarrow [0, \infty]$ given by

$$Q'\zeta := \begin{cases} Q(\zeta, \zeta) & \zeta \in D(Q) \\ \infty & \text{else.} \end{cases}$$

closed,
densely-defined
quadratic forms

$$\|A^{1/2}\cdot\|^2$$



generally
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selfadjoint operators

$$A$$



C_0 -semigroups of
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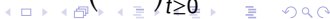
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Dirichlet forms and Markov semigroups

(X, m) – positive measure space

Definition

A map $S : L^2(X, m) \rightarrow L^2(X, m)$ is **Markov** if for all $f \in L^2(X, m)$,

$$0 \leq f \leq 1 \implies 0 \leq Sf \leq 1.$$

Definition (Based on Beurling–Deny, Acta Math., 1958)

A **Dirichlet form** on (X, m) is a closed, densely defined, quadratic form Q on $L^2(X, m)$ such that for all \mathbb{R} -valued $f \in L^2(X, m)$,

$$Q(\min(\max(f, 0), 1)) \leq Q(f).$$

Theorem (Beurling–Deny)

Dirichlet forms
on (X, m)



symmetric Markov semigroups
on $L^2(X, m) / L^\infty(X, m)$

Dirichlet forms and Markov semigroups

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Non-commutative Dirichlet forms and Markov semigroups

- ★ **Commutative** = “classical” = on a **positive measure space** with a **reference measure** (Beurling–Deny, 1958).
- ★ **Non-commutative** = on a **von Neumann algebra** with a **reference weight**.
 - ▶ We use the general definition of Goldstein–Lindsay (Math. Ann., 1999).

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Main result

\mathbb{G} – locally compact quantum group

(Recall: $L^\infty(\mathbb{G})$ – underlying von Neumann algebra, $L^2(\mathbb{G})$ – a standard Hilbert space; φ – left Haar weight.)

Theorem (Skalski–V)

There exist 1 – 1 correspondences between:

- 1 w^* -continuous, symmetric, **convolution semigroups** of states on \mathbb{G} ;
- 2 completely **Dirichlet forms** w.r.t. φ that are right-translation invariant;
- 3 completely **Markov semigroups** on $L^2(\mathbb{G})$ that are symmetric and contained in $L^\infty(\hat{\mathbb{G}})$;
- 4 completely **Markov semigroups** on $L^\infty(\mathbb{G})$ that are right-translation invariant and KMS-symmetric w.r.t. φ .

Earlier results

Our main theorem is definitive.

It unifies and extends the **classical** and the **compact** cases:

- $G = G$:



- $G = \hat{G}$:

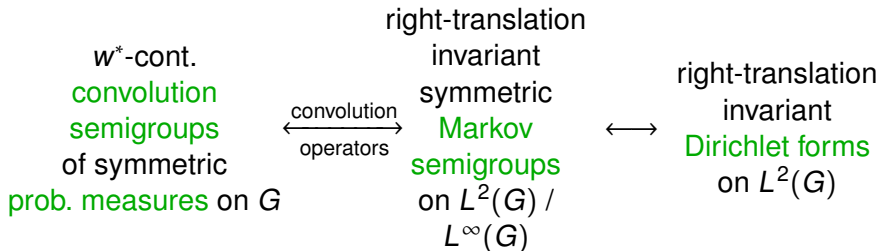


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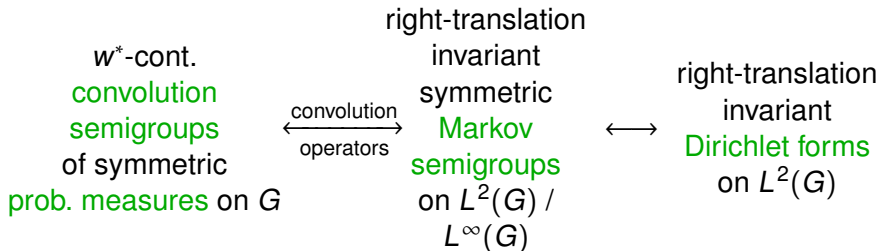


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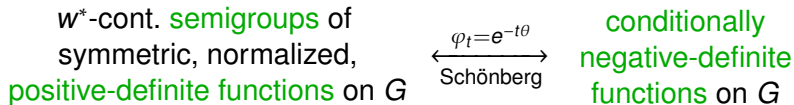
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- \mathbb{G} – **compact** quantum group: Cipriani–Franz–Kula (JFA, 2014).
Such \mathbb{G} has a canonical Hopf $*$ -algebra that is contained in the domain of all Dirichlet forms \rightsquigarrow the problem becomes algebraic.

Detour: conditionally negative-definite functions

Definition

A continuous $\theta : G \rightarrow \mathbb{R}$ is **conditionally negative definite** if:

- 1 $\theta(e) = 0$;
- 2 $\theta(g^{-1}) = \theta(g)$ for all $g \in G$;
- 3 $(\theta(g_i) + \theta(g_j) - \theta(g_j^{-1}g_i))_{1 \leq i, j \leq n}$ is positive definite for all $n \in \mathbb{N}$ and $g_1, \dots, g_n \in G$.

Schönberg's Theorem

A continuous $\theta : G \rightarrow \mathbb{R}$ satisfying 1 and 2
is **CND** $\iff e^{-t\theta}$ is positive definite for all $t \geq 0$.

Detour: conditionally negative-definite functions

Example

For $n \in \mathbb{N}$ and $0 \leq \alpha \leq 2$, the function $\mathbb{R}^n \rightarrow [0, \infty)$ given by $x \mapsto \|x\|^\alpha$ is conditionally negative definite.

Example (Haagerup, Invent. Math., 1978/79)

Let $n \in \mathbb{N}$. The function $\mathbb{F}_n \rightarrow [0, \infty)$ given by

$$s \mapsto |s|$$

is conditionally negative definite.

\rightsquigarrow Various applications.

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Approximation properties for groups

Geometric characterizations

Theorem (Guichardet, '72 + Delorme, '77; Akemann–Walter, '81)

Assume that G is σ -compact.

- 1 G does **not** have **property (T)**
 \iff it has an **unbounded** conditionally negative-definite function
 \iff it has a w^* -cont. semigroup of symm. normalized pos.-def. functions that is **not norm continuous**.
- 2 G has the **Haagerup property**
 \iff it has a **proper** conditionally negative-definite function
 \iff it has a w^* -cont. semigroup of symm. normalized pos.-def. functions that is **C_0 in positive time**.

(proper = goes to ∞ at ∞).

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The CND function $s \mapsto |s|$ on \mathbb{F}_n , $n \in \mathbb{N}$.

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Approximation properties for quantum groups

Geometric characterizations: the discrete case

\mathbb{G} – second countable, **discrete** quantum group.

By virtue of its special structure, one can define CND functions on \mathbb{G} .

Theorem (Kyed, JFA, 2011)

If \mathbb{G} is unimodular, then it does **not** have **property (T)**

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Theorem (Daws–Fima–Skalski–White, Crelle’s Journal, 2016)

\mathbb{G} has the **Haagerup property**

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Again, the special structure allows for more algebraic proofs. One important component is a version of Schönberg’s Theorem for finite-dimensional co-algebras due to Schürmann (1985).

- Many applications, e.g. Daws–Skalski–V, Comm. Math. Phys., 2017.

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- Many applications, e.g. Daws–Skalski–V, Comm. Math. Phys., 2017.

Approximation properties for quantum groups

Geometric characterizations: the discrete case

\mathbb{G} – second countable, **discrete** quantum group.

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Theorem (Kyed, JFA, 2011)

If \mathbb{G} is unimodular, then it does **not** have **property (T)**

\iff it has an **unbounded** conditionally negative-definite function.

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Approximation properties for quantum groups

Geometric characterizations: the general case

\mathbb{G} – second countable, locally compact quantum group.

Theorems (Skalski–V)

- \mathbb{G} does **not** have **property (T)**
 - $\iff \hat{\mathbb{G}}$ has a completely Dirichlet form w.r.t. $\hat{\varphi}$ that is right-translation invariant and **unbounded**
 - $\iff \hat{\mathbb{G}}$ has a w^* -continuous, symmetric, convolution semigroup of states that is **not norm continuous**.
- \mathbb{G} has the **Haagerup property**
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 - $\iff \hat{\mathbb{G}}$ has a w^* -continuous, symmetric, convolution semigroup of states that is **C_0 in positive time**.

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Thank you for your attention!