Convolution semigroups on quantum groups and non-commutative Dirichlet forms

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Semigroups of Operators: Theory and Applications
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Operator algebras

A C^*-algebra is...

an involutive Banach algebra such that $\|x^*x\| = \|x\|^2$ for all x.

- Commutative case: $C_0(\Omega)$

A von Neumann algebra is...

a $*$-subalgebra of $B(\mathcal{H})$, containing I and SOT-closed.

- Commutative case: $L^\infty(X, \mathbb{A}, \mu)$ over $\mathcal{H} = L^2(X, \mathbb{A}, \mu)$

A state of a C^*-algebra $\mathcal{A}$ is a functional on $\mathcal{A}$ that maps positive elements into $[0, \infty)$ and has norm 1.

Example

States of $C_0(\Omega)$ $\longleftrightarrow$ regular probability measures on Ω.

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A group as a quantum group

G – locally compact group.

1. $L^\infty(G)$ – a von Neumann algebra.
2. Co-multiplication: the $*$-homomorphism

$$\Delta : L^\infty(G) \to L^\infty(G) \,\overline{\otimes}\, L^\infty(G) \cong L^\infty(G \times G)$$

   defined by

$$(\Delta(f))(t,s) := f(ts) \qquad (f \in L^\infty(G)).$$

   By associativity, we have $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

3. Left and right Haar measures. View them as functions $\varphi, \psi : L^\infty(G)_+ \to [0,\infty]$ by $\varphi(f) := \int_G f(t)\,\mathrm{d}t_\ell$, $\psi(f) := \int_G f(t)\,\mathrm{d}t_r$.

Motivation for quantum groups

Lack of Pontryagin duality for non-abelian l.c. groups.

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Locally compact quantum groups

Definition (Kustermans–Vaes, '00)

A locally compact quantum group is a pair $\mathbb{G} = (M, \Delta)$ such that:

1. M is a von Neumann algebra
2. $\Delta : M \to M \overline{\otimes} M$ is a co-multiplication: a normal, faithful, unital $*$-homomorphism which is co-associative, i.e.,

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

3. There are two n.s.f. weights φ, ψ on M (the Haar weights) with:
   - $\varphi((\omega \otimes \mathrm{id})\Delta(x)) = \omega(\mathbb{1})\varphi(x)$ when $\omega \in M_*^+$, $x \in M^+$ and $\varphi(x) < \infty$
   - $\psi((\mathrm{id} \otimes \omega)\Delta(x)) = \omega(\mathbb{1})\psi(x)$ when $\omega \in M_*^+$, $x \in M^+$ and $\psi(x) < \infty$.

Denote $L^\infty(\mathbb{G}) := M$.
Have it act standardly on the Hilbert space $L^2(\mathbb{G})$.

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Locally compact quantum groups

Features

Duality $\mathbb{G} \mapsto \hat{\mathbb{G}}$ within the category satisfying $\hat{\hat{\mathbb{G}}} = \mathbb{G}$.

Three "faces" (algebras): the von Neumann algebra $L^\infty(\mathbb{G})$, the "reduced" C^*-algebra $C_0(\mathbb{G})$, and the "universal" C^*-algebra $C_0^{\mathrm{u}}(\mathbb{G})$.

The antipode: an (unbounded) operator on $L^\infty(\mathbb{G})$ / $C_0(\mathbb{G})$ / $C_0^{\mathrm{u}}(\mathbb{G})$. Decomposes into:

- a "bounded part": the unitary antipode, an anti-automorphism;
- an "unbounded part": the scaling group.

Two basic examples

- $\mathbb{G} = G$
- $\mathbb{G} = \hat{G}$

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Convolution semigroups

$\mathbb{G}$ – locally compact quantum group.

The conjugate space $C_0^{\mathrm{u}}(\mathbb{G})^*$ carries a convolution $\star$ turning it into a Banach algebra with unit ϵ (the co-unit).

Definition

A convolution semigroup of states on $\mathbb{G}$ is a family $(\mu_t)_{t\geq 0}$ of states of $C_0^{\mathrm{u}}(\mathbb{G})$ such that

$$\mu_0 = \epsilon \quad \text{and} \quad \mu_s \star \mu_t = \mu_{s+t} \qquad (\forall s, t \geq 0).$$

Adjectives:

- w^*-continuous
- symmetric = invariant under the unitary antipode.

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Convolution semigroups

Examples

of w^*-continuous, symmetric, convolution semigroups of states on $\mathbb{G}$:

1. $\mathbb{G} = G$: w^*-continuous, symmetric, convolution semigroups of probability measures on G;

   that is:

   families $(\mu_t)_{t \geq 0}$ of probability Borel measures on G satisfying

   $$\mu_0 = \delta_e \quad \text{and} \quad \mu_s \star \mu_t = \mu_{s+t} \qquad (\forall s, t \geq 0)$$

   that are w^*-continuous and invariant under “inversion of sets”.

Convolution semigroups

Examples

of w^*-continuous, symmetric, convolution semigroups of states on $\mathbb{G}$:

2. $\mathbb{G} = \hat{G}$: w^*-continuous, symmetric, semigroups of normalized positive-definite functions on G;

   that is:

   families $(\varphi_t)_{t\geq 0}$ of normalized positive-definite functions on G satisfying

   $$\varphi_0 \equiv 1 \quad \text{and} \quad \varphi_s \cdot \varphi_t = \varphi_{s+t} \qquad (\forall s, t \geq 0)$$

   that are w^*-continuous and invariant under inversion (= real).

Dirichlet forms and Markov semigroups

Definition

A (non-negative) quadratic form on a Hilbert space $\mathcal{H}$ is a semi-inner product $Q : D(Q) \times D(Q) \to \mathbb{C}$ on a subspace $D(Q)$ of $\mathcal{H}$.

- Densely defined if $D(Q)$ is dense in $\mathcal{H}$.
- Closedness.

More convenient to work with $Q' : \mathcal{H} \to [0, \infty]$ given by

$$Q'\zeta := \begin{cases} Q(\zeta, \zeta) & \zeta \in D(Q) \\ \infty & \text{else.} \end{cases}$$

closed, densely-defined quadratic forms $\left\|A^{1/2}\cdot\right\|^2$	$\longleftrightarrow$	generally unbounded, positive selfadjoint operators A	$\longleftrightarrow$	C_0-semigroups of selfadjoint contractions $\left(e^{-tA}\right)_{t\geq 0}$

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Dirichlet forms and Markov semigroups

(X, m) – positive measure space

Definition

A map $S : L^2(X, m) \to L^2(X, m)$ is Markov if for all $f \in L^2(X, m)$,

$$0 \leq f \leq 1 \implies 0 \leq Sf \leq 1.$$

Definition (Based on Beurling–Deny, Acta Math., 1958)

A Dirichlet form on (X, m) is a closed, densely defined, quadratic form Q on $L^2(X, m)$ such that for all $\mathbb{R}$-valued $f \in L^2(X, m)$,

$$Q\Big(\min\left(\max(f, 0), 1\right)\Big) \leq Q(f).$$

Theorem (Beurling–Deny)

Dirichlet forms on (X, m) $\longleftrightarrow$ symmetric Markov semigroups on $L^2(X, m)$ / $L^\infty(X, m)$

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Non-commutative Dirichlet forms and Markov semigroups

- ⋆ Commutative = "classical" = on a positive measure space with a reference measure (Beurling–Deny, 1958).
- ⋆ Non-commutative = on a von Neumann algebra with a reference weight.
  - ▶ We use the general definition of Goldstein–Lindsay (Math. Ann., 1999).

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Main result

$\mathbb{G}$ – locally compact quantum group

(Recall: $L^\infty(\mathbb{G})$ – underlying von Neumann algebra, $L^2(\mathbb{G})$ – a standard Hilbert space; φ – left Haar weight.)

Theorem (Skalski–V)

There exist 1 – 1 correspondences between:

1. *w^*-continuous, symmetric, convolution semigroups of states on $\mathbb{G}$;*
2. *completely Dirichlet forms w.r.t. φ that are right-translation invariant;*
3. *completely Markov semigroups on $L^2(\mathbb{G})$ that are symmetric and contained in $L^\infty(\hat{\mathbb{G}})$;*
4. *completely Markov semigroups on $L^\infty(\mathbb{G})$ that are right-translation invariant and KMS-symmetric w.r.t. φ.*

Earlier results

Our main theorem is definitive.
It unifies and extends the classical and the compact cases:

- $\mathbb{G} = G$:

  w^*-cont. convolution semigroups of symmetric prob. measures on G $\xleftrightarrow[\text{operators}]{\text{convolution}}$ right-translation invariant symmetric Markov semigroups on $L^2(G)$ / $L^\infty(G)$ $\longleftrightarrow$ right-translation invariant Dirichlet forms on $L^2(G)$

- $\mathbb{G} = \hat{G}$:

  w^*-cont. semigroups of symmetric, normalized, positive-definite functions on G $\xleftrightarrow[\text{Schönberg}]{\varphi_t = e^{-t\theta}}$ conditionally negative-definite functions on G

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Earlier results

- $\mathbb{G}$ – compact quantum group: Cipriani–Franz–Kula (JFA, 2014).

  Such $\mathbb{G}$ has a canonical Hopf $*$-algebra that is contained in the domain of all Dirichlet forms $\rightsquigarrow$ the problem becomes algebraic.

Detour: conditionally negative-definite functions

Definition

A continuous $\theta : G \to \mathbb{R}$ is conditionally negative definite if:

1. $\theta(e) = 0$;
2. $\theta(g^{-1}) = \theta(g)$ for all $g \in G$;
3. $\left(\theta(g_i) + \theta(g_j) - \theta(g_j^{-1} g_i)\right)_{1 \le i,j \le n}$ is positive definite for all $n \in \mathbb{N}$ and $g_1, \ldots, g_n \in G$.

Schönberg's Theorem

A continuous $\theta : G \to \mathbb{R}$ satisfying 1 and 2

is CND $\iff$ $e^{-t\theta}$ is positive definite for all $t \geq 0$.

Detour: conditionally negative-definite functions

Example

For $n \in \mathbb{N}$ and $0 \le \alpha \le 2$, the function $\mathbb{R}^n \to [0, \infty)$ given by $x \mapsto \|x\|^\alpha$ is conditionally negative definite.

Example (Haagerup, Invent. Math., 1978/79)

Let $n \in \mathbb{N}$. The function $\mathbb{F}_n \to [0, \infty)$ given by

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Theorem (Guichardet, '72 + Delorme, '77; Akemann–Walter, '81)

Assume that G is σ-compact.

1. *G does not have property (T)*
   $\iff$ *it has an unbounded conditionally negative-definite function*
   $\iff$ *it has a w^*-cont. semigroup of symm. normalized pos.-def. functions that is not norm continuous.*
2. *G has the Haagerup property*
   $\iff$ *it has a proper conditionally negative-definite function*
   $\iff$ *it has a w^*-cont. semigroup of symm. normalized pos.-def. functions that is C_0 in positive time.*

(proper = goes to ∞ at ∞).

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Approximation properties for quantum groups

Geometric characterizations: the discrete case

$\mathbb{G}$ – second countable, discrete quantum group.
By virtue of its special structure, one can define CND functions on $\mathbb{G}$.

Theorem (Kyed, JFA, 2011)

If $\mathbb{G}$ is unimodular, then it does not have property (T)
$\iff$ it has an unbounded conditionally negative-definite function.

Theorem (Daws–Fima–Skalski–White, Crelle's Journal, 2016)

$\mathbb{G}$ has the Haagerup property
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Again, the special structure allows for more algebraic proofs. One important component is a version of Schönberg's Theorem for finite-dimensional co-algebras due to Schürmann (1985).

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$\mathbb{G}$ – second countable, discrete quantum group.
By virtue of its special structure, one can define CND functions on $\mathbb{G}$.

Theorem (Kyed, JFA, 2011)

If $\mathbb{G}$ is unimodular, then it does not have property (T)
$\iff$ it has an unbounded conditionally negative-definite function.

Theorem (Daws–Fima–Skalski–White, Crelle's Journal, 2016)

$\mathbb{G}$ has the Haagerup property
$\iff$ it has a proper conditionally negative-definite function.

Again, the special structure allows for more algebraic proofs. One important component is a version of Schönberg's Theorem for finite-dimensional co-algebras due to Schürmann (1985).

- Many applications, e.g. Daws–Skalski–V, Comm. Math. Phys., 2017.

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$\mathbb{G}$ – second countable, locally compact quantum group.

Theorems (Skalski–V)

1. $\mathbb{G}$ does not have property (T)
   $\iff \hat{\mathbb{G}}$ has a completely Dirichlet form w.r.t. $\hat{\varphi}$ that is right-translation invariant and unbounded
   $\iff \hat{\mathbb{G}}$ has a w^*-continuous, symmetric, convolution semigroup of states that is not norm continuous.
2. $\mathbb{G}$ has the Haagerup property
   $\iff \hat{\mathbb{G}}$ has a completely Dirichlet form w.r.t. $\hat{\varphi}$ that is right-translation invariant and proper
   $\iff \hat{\mathbb{G}}$ has a w^*-continuous, symmetric, convolution semigroup of states that is C_0 in positive time.

A. Skalski and A. Viselter, *Convolution semigroups on locally compact quantum groups and noncommutative Dirichlet forms*
Journal de Mathématiques Pures et Appliquées, to appear, 47 pp.

M. Daws, A. Skalski and A. Viselter, *Around Property (T) for quantum groups*
Comm. Math. Phys. 353 (2017), no. 1, 69–118.

M. Daws, P. Fima, A. Skalski and S. White, *The Haagerup property for locally compact quantum groups*
J. Reine Angew. Math. 711 (2016), 189–229.

F. Cipriani, U. Franz and A. Kula, *Symmetries of Lévy processes on compact quantum groups, their Markov semigroups and potential*
J. Funct. Anal. 266 (2014), no. 5, 2789–2844.

D. Kyed, *A cohomological description of property (T) for quantum groups*
J. Funct. Anal. 261 (2011), no. 6, 1469–1493.

S. Goldstein and J. M. Lindsay, *Markov semigroups KMS-symmetric for a weight*
Math. Ann. 313 (1999), no. 1, 39–67.

C. A. Akemann and M. E. Walter, *Unbounded negative definite functions*
Canad. J. Math. 33 (1981), no. 4, 862–871.

U. Haagerup, *An example of a non nuclear C^*-algebra, which has the metric approximation property*
Invent. Math. 50 (1978/79), no. 3, 279–293.

Thank you for your attention!