Isometry Lie algebras of indefinite homogeneous spaces of finite volume

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IX International Meeting on Lorentzian Geometry
IMPAN, June 2018

I Geometric background

(M, g) is a connected pseudo-Riemannian homogeneous manifold of finite volume.

- The metric tensor g is non-degenerate but can be indefinite.
- A proper subspace $U \subset T_pM$ can be totally isotropic, that is, $\mathrm{g}_p|_U = 0$.
- The index s of (M, g) is the maximal dimension of a totally isotropic subspace $U \subset T_pM$.
  - Riemannian $s = 0$ (positive definite).
  - Lorentzian $s = 1$ (“lightlike lines”).

(M, g) is a connected pseudo-Riemannian homogeneous manifold of finite volume;

- $M = G/H$ for a connected Lie group G and a closed subgroup H,
- G acts transitively and by isometries (in particular volume-preserving),
- G acts almost effectively (H has no connected normal subgroups).

Question:

- Which Lie groups G can be isometry groups of such M?
- Which subgroups $H \subset G$ can be stabilizers of such actions?
- How is geometry of G and M related?

Previous work

- Zimmer's and Gromov's work in the 1980s on rigid geometric structures.
- Adams & Stuck (1997), Zeghib (1998):
  Classification of isometry groups of compact Lorentzian manifolds.
  (Higher indices are much more difficult.)
- Zeghib (1998):
  Classification of compact homogeneous Lorentzian manifolds.

Assumptions:

- G acts transitively and by isometries on $M = G/H$,
- G acts almost effectively (H has no connected normal subgroups).

The metric g on M induces a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Then:

- $\langle \cdot, \cdot \rangle$ is $\mathrm{Ad}_{\mathfrak{g}}(H)$-invariant (and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-invariant),

$$\langle \mathrm{Ad}_{\mathfrak{g}}(h)x, \mathrm{Ad}_{\mathfrak{g}}(h)y \rangle = \langle x, y \rangle \quad \text{for all } h \in H,$$
$$\langle \mathrm{ad}_{\mathfrak{g}}(h')x, y \rangle = -\langle x, \mathrm{ad}_{\mathfrak{g}}(h')y \rangle \quad \text{for all } h' \in \mathfrak{h}.$$

- The kernel of $\langle \cdot, \cdot \rangle$ is

$$\mathfrak{g}^{\perp} = \{x \in \mathfrak{g} \mid \langle x, \cdot \rangle = 0\} = \mathfrak{h}.$$

$\mathsf{G} \leq \mathrm{GL}_n(\mathbb{C})$ is a linear algebraic group (given by polynomial equations).
For a subgroup of $H \leq \mathsf{G}$, let $\overline{H}^{\mathrm{z}}$ denote the Zariski closure of H in G;

- $\overline{H}^{\mathrm{z}}$ is the smallest algebraic subgroup of G that contains H.
- H is Zariski-dense in G if $\overline{H}^{\mathrm{z}} = \mathsf{G}$.

Examples
$\mathbb{Z}$ is Zariski-dense in $\mathbb{C}$ (or $\mathbb{R}$),
$\mathrm{SL}_n(\mathbb{Z})$ is Zariski-dense in $\mathrm{SL}_n(\mathbb{C})$ (or $\mathrm{SL}_n(\mathbb{R})$),...

Invariance under unipotent operators

Let (M, g) be a pseudo-Riemannian manifold of finite volume and $G \subseteq \mathrm{Iso}(M, g)$. The adjoint representation of G on $\mathfrak{g}$ induces a representation $\varrho(G)$ on $\mathrm{Sym}^2\mathfrak{g}^*$.

Invariance Theorem

For any $p \in M$, the symmetric bilinear form s_p on $\mathfrak{g}$ given by

$$s_p(x, y) = g_p(X_p, Y_p),$$

is invariant by all unipotent elements in the Zariski closure $\overline{\varrho(G)}^{z}$ in $\mathrm{GL}(\mathrm{Sym}^2\mathfrak{g}^*)$. (Here X, Y denote the Killing fields on M corresponding to $x, y \in \mathfrak{g}$.)

Proof

- Set $V = \mathrm{Sym}^2\mathfrak{g}^*$ and $\pi : V \to \mathbb{P}(V)$ projectivization.
- Finite G-invariant measure on M induces finite G-invariant measure μ on $\mathbb{P}(V)$.
- Furstenberg Lemma: supp μ is finite union of projective subspaces $\pi(W_j)$, and $\mathrm{PGL}(V)_\mu$ restricted to $\pi(W_j)$ has compact closure.
- Pick j such that $s_p \in W_j$.
- $\mathrm{PGL}(V)_\mu$ is real algebraic, so contains $\pi_*(\overline{\varrho(G)}^{z})$.
- If $u \in \overline{\varrho(G)}^{z}$ unipotent, then $\pi_*(u)|_{\pi(W_j)}$ is unipotent and contained in a compact group, hence trivial. □

Recall:

- G acts transitively and by isometries on $M = G/H$,
- G acts almost effectively on M.

Apply Invariance Theorem to homogeneous M and $\langle\cdot,\cdot\rangle$:

- $\langle\cdot,\cdot\rangle$ is invariant by all unipotent elements in $\overline{\mathrm{Ad}_{\mathfrak{g}}(G)}^{\mathrm{Z}}$.
- In particular, all nilpotent elements in of $\mathfrak{Lie}(\overline{\mathrm{Ad}_{\mathfrak{g}}(G)}^{\mathrm{Z}})$ are skew-symmetric with respect to $\langle\cdot,\cdot\rangle$.
- We call $\langle\cdot,\cdot\rangle$ nil-invariant.

Let KS be a Levi subgroup of G, where

- K is compact semisimple and
- S is semisimple without compact factors.

Let R denote the solvable radical of G, so that $G = KSR$, and let N denote the nilradical of R.

Corollary

$\langle\cdot,\cdot\rangle$ *is invariant by* $\mathrm{Ad}_{\mathfrak{g}}(S)$ *and* $\mathrm{Ad}_{\mathfrak{g}}(N)$.

II Compact indefinite solvmanifolds

For now, assume that G is a connected solvable Lie group,
so that (M, g) is a compact pseudo-Riemannian solvmanifold.

- To understand G, study solvable Lie algebras $\mathfrak{g}$ with nil-invariant $\langle \cdot, \cdot \rangle$.
- Recall: $\mathfrak{g}^{\perp} = \mathfrak{h}$.
- If G acts almost effectively, then $\mathfrak{g}^{\perp}$ contains no ideal $\neq \mathbf{0}$ in $\mathfrak{g}$.

A solvable Lie algebra with Lorentzian (nil-)invariant product is the oscillator algebra

$$\mathfrak{g} = \mathfrak{osc}(\alpha) = \mathbb{R} \ltimes \mathfrak{hei}_{2n+1}$$

where the Heisenberg algebra is

$$\mathfrak{hei}_{2n+1} = \mathbb{R}^{2n} \times \mathbb{R},$$

two-step nilpotent with one-dimensional center $\mathbb{R}$,
and $\mathbb{R}$ acts on $\mathbb{R}^{2n}$ by rotations with weights $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Define Lorentzian $\langle \cdot, \cdot \rangle$ on $\mathfrak{osc}(\alpha)$ by a definite scalar product on $\mathbb{R}^{2n}$ and a dual pairing of totally isotropic subspaces $\mathbb{R}$ and $\mathbb{R}$.

- Medina (1985) and Hilgert & Hofmann (1985) showed that this is the only solvable non-abelian Lie algebra with invariant Lorentzian product.
- For index ≥ 2, Kath and Olbrich (2004) gave a classification scheme for Lie algebras with invariant scalar product, and a classification for index 2.
- For index ≥ 3 classification becomes extremely complicated.

Proposition

Let $\langle \cdot, \cdot \rangle$ be a symmetric nil-invariant bilinear form on a solvable Lie algebra $\mathfrak{g}$.

- *Then $\mathfrak{g}^{\perp}$ is an ideal in $\mathfrak{g}$.*
- *If $\mathfrak{g}^{\perp}$ contains no non-trivial ideals of $\mathfrak{g}$, then $\mathfrak{g}^{\perp} = \mathbf{0}$.*

Theorem A

Assume G is solvable.
Then G acts almost freely on M.

Meaning:

- The stabiliser $\Gamma = G_x$ is discrete.
- The metric g on M pulls back to a left-invariant pseudo-Riemannian metric g_G on G.
- g_G is invariant under conjugation by Γ.

$\mathfrak{g}$ is solvable with nil-invariant scalar product $\langle\cdot,\cdot\rangle$.

Proposition

$\langle\cdot,\cdot\rangle$ is invariant on $\mathfrak{g}$.

Proof

- Let $\mathfrak{j}$ be a totally isotropic central ideal in $\mathfrak{g}$. Define the reduction $\overline{\mathfrak{g}} = \mathfrak{j}^{\perp}/\mathfrak{j}$.
- $\langle\cdot,\cdot\rangle$ induces a nil-invariant scalar product $\langle\cdot,\cdot\rangle_{\overline{\mathfrak{g}}}$ on $\overline{\mathfrak{g}}$.
- Assume that $\langle\cdot,\cdot\rangle_{\overline{\mathfrak{g}}}$ is invariant on $\overline{\mathfrak{g}}$.
- Verify that $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ is invariant if $\langle\cdot,\cdot\rangle_{\overline{\mathfrak{g}}}$ is.
- Iterated reduction to the abelian case and induction yield the result. □

Theorem B

Assume G is solvable.
Then g pulls back to a bi-invariant pseudo-Riemannian metric g_G on G.

Corollary A+B

The universal cover $\widetilde{M}$ of M is a pseudo-Riemannian symmetric space.
In particular, M is locally symmetric.

Theorem C

Assume G is solvable and effective.
Then $G = \mathrm{Iso}(M)^{\circ}$.

Implications:

- Johnson (1972) showed that any solvmanifold has presentations by connected solvable groups of arbitrary dimension.
- Theorem A shows that most of them cannot act isometrically.
- Theorem C shows that no larger non-solvable group can act isometrically.

III Compact indefinite homogeneous spaces for arbitrary Lie groups

Now let $G = KSR$ be an arbitrary connected Lie group, and as before,

- K compact semisimple,
- S semisimple without compact factors,
- R the solvable radical of G.

We study Lie algebras

$$\mathfrak{g} = (\mathfrak{k} \times \mathfrak{s}) \ltimes \mathfrak{r}$$

with nil-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$.

- $\langle \cdot, \cdot \rangle$ is a priori invariant under $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{s})$ and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{n})$ (nilradical).
- From the solvable case it follows that *the restriction* $\langle \cdot, \cdot \rangle_{\mathfrak{s} \ltimes \mathfrak{r}}$ *is invariant under* $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{s} \ltimes \mathfrak{r})$.

If the index of $\langle\cdot,\cdot\rangle$ is 1, then $\mathfrak{g} = \mathfrak{k} \times \mathfrak{s} \times \mathfrak{r}$, with either $\mathfrak{s} = \mathbf{0}$ or $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{R})$.

Zeghib's Theorem (1998)

Let M be a homogeneous Lorentzian manifold of finite volume.
Then, up to "Riemannian type factors", M is one of the following:

1. $\mathrm{Iso}(M, \mathrm{g})$ *contains a cover of* $\mathrm{PSL}_2(\mathbb{R})$.
   $\widetilde{M} = \widetilde{\mathrm{SL}_2}(\mathbb{R}) \times L$, *where L is a compact Riemannian homogeneous space.*
2. $\mathrm{Iso}(M, \mathrm{g})$ *contains an oscillator group* $\mathrm{Osc}(\alpha)$.
   $\widetilde{M} = \mathrm{Osc}(\alpha) \times_{\mathbf{S}^1} L$, *and* $M = \widetilde{M}/\Gamma$, *where Γ is isomorphic to a lattice in* $\mathrm{Osc}(\alpha)$.

Suppose the index of $\langle\cdot,\cdot\rangle$ is ≥ 1.

Theorem D

1. $\langle\cdot,\cdot\rangle$ *is* $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{s}\ltimes\mathfrak{r})$*-invariant.*
2. $\langle\cdot,\cdot\rangle_{\mathfrak{s}\ltimes\mathfrak{r}}$ *is* $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$*-invariant.*

Proof

- Decompose $\mathfrak{g} = (\mathfrak{s}\ltimes\mathfrak{r}) + \mathfrak{g}^{\mathfrak{s}}$, where $[\mathfrak{s},\mathfrak{g}^{\mathfrak{s}}] = \mathbf{0}$.
- Apply result for solvable case to subalgebra $\mathbb{R}x + \mathfrak{r}$ for $x \in \mathfrak{g}^{\mathfrak{s}}$. This shows part 1.
- We know $\langle\cdot,\cdot\rangle$ is $\mathrm{ad}(\mathfrak{s}\ltimes\mathfrak{n})$-invariant. Nil-invariance relations show that it only remains to show that $\mathrm{ad}(x)$ is skew-symmetric in case $[x,\mathfrak{k}] = \mathbf{0}$.
- Then $\langle[x,\mathfrak{k}],\mathfrak{g}\rangle = 0 = -\langle\mathfrak{k},[x,\mathfrak{g}]\rangle$, since $x \perp \mathfrak{k}$ (follows from semisimplicity of $\mathfrak{k}$). This shows part 2. □

Note: We do not assume $\mathfrak{g}^{\perp}$ contains no non-zero ideals.

Theorem E

If the index of $\langle \cdot, \cdot \rangle$ is ≤ 2 and $\mathfrak{g}^{\perp}$ does not contain a non-trivial ideal of $\mathfrak{g}$, then:

1. $\mathfrak{g}$ *is a direct sum of ideals* $\mathfrak{g} = \mathfrak{k} \times \mathfrak{s} \times \mathfrak{r}$.
2. $\mathfrak{g}^{\perp} \subset \mathfrak{z}(\mathfrak{r}) \times \mathfrak{k}$ *and* $\mathfrak{g}^{\perp} \cap (\mathfrak{s} \times \mathfrak{r}) = \mathbf{0}$.

Theorem F

Let $\mathfrak{g}$ be a Lie algebra with nil-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ of index 2, and assume that $\mathfrak{g}^{\perp}$ does not contain a non-trivial ideal of $\mathfrak{g}$.
Then one of the following cases occurs:

(I) $\mathfrak{g} = \mathfrak{r} \times \mathfrak{k}$, *where $\mathfrak{r}$ is one of the following:*

- (a) $\mathfrak{r}$ *is abelian.*
- (b) $\mathfrak{r}$ *is Lorentzian of oscillator type.*
- (c) $\mathfrak{r}$ *is solvable but non-abelian with invariant scalar product of index 2.*

(II) $\mathfrak{g} = \mathfrak{a} \times \mathfrak{k} \times \mathfrak{s}$. *Here, $\mathfrak{a}$ is abelian, $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$ with a non-degenerate invariant scalar product of index 2. Moreover, $\mathfrak{a}$ is definite and $(\mathfrak{a} \times \mathfrak{k}) \perp \mathfrak{s}$.*

(III) $\mathfrak{g} = \mathfrak{r} \times \mathfrak{k} \times \mathfrak{sl}_2(\mathbb{R})$, *where $\mathfrak{sl}_2(\mathbb{R})$ is Lorentzian, $(\mathfrak{r} \times \mathfrak{k}) \perp \mathfrak{sl}_2(\mathbb{R})$, and $\mathfrak{r}$ is one of the following:*

- (a) $\mathfrak{r}$ *is abelian and either semidefinite or Lorentzian.*
- (b) $\mathfrak{r}$ *is Lorentzian of oscillator type.*

Example

- Let $\mathfrak{k} = \mathfrak{so}_3$, and let $\mathfrak{r} = \mathfrak{so}_3^{\mathrm{I}} \oplus \mathfrak{so}_3^{\mathrm{II}}$, considered as a vector space.
- Let $\mathfrak{so}_3^{\triangle}$ be the diagonal embedding of $\mathfrak{so}_3$ in $\mathfrak{r}$.
- Let $k \in \mathfrak{k}$ act on $x = x_1^{\mathrm{I}} + x_2^{\mathrm{II}} \in \mathfrak{r}$ by

$$\mathrm{ad}(k)x = [k, x_1]^{\mathrm{I}}.$$

  This makes $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{r}$ into a Lie algebra with solvable radical $\mathfrak{r}$.
- Let κ denote the Killing form on $\mathfrak{so}_3$. We define $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by

$$\langle k, x_1^{\mathrm{I}} + x_2^{\mathrm{II}} \rangle = \kappa(k, x_1) - \kappa(k, x_2), \quad \mathfrak{k} \perp \mathfrak{k}, \quad \mathfrak{r} \perp \mathfrak{r}$$

  for all $k \in \mathfrak{k}$, $x_1^{\mathrm{I}} + x_2^{\mathrm{II}} \in \mathfrak{so}_3$.
- We can verify that $\langle \cdot, \cdot \rangle$ is $\mathrm{ad}(\mathfrak{r})$-invariant, so $\langle \cdot, \cdot \rangle$ is a nil-invariant form on $\mathfrak{g}$.
- Since $\mathfrak{r}^{\perp} = \mathfrak{r}$,

$$\mathfrak{g}^{\perp} = \mathfrak{k}^{\perp} \cap \mathfrak{r} = \mathfrak{so}_3^{\triangle} \subset \mathfrak{r} \quad \text{and} \quad \mathfrak{r}^{\perp} \cap \mathfrak{r} = \mathfrak{r} = \mathfrak{so}_3^{\mathrm{I}} \oplus \mathfrak{so}_3^{\mathrm{II}}.$$

  In particular, the index on $\mathfrak{g}/\mathfrak{g}^{\perp}$ is 3.

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