Lorentzian geometry and CR structures

Arman TAGHAVI-CHABERT

American University of Beirut

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Motivation

$(\mathcal{M}, [g])$ 4-dim conformal Lorentzian manifold
equipped with shearfree congruence of null geodesics

$\Big\updownarrow$ Involutive totally null complex 2-plane distribution

$(\underline{\mathcal{M}}, \underline{H}, \underline{J})$ 3-dim CR manifold

Curvature condition on the Weyl tensor W

generator k of congruence: Principal null direction $W(k, \cdot, k, \cdot) = 0 \mod \kappa \odot \alpha$

$\kappa = g(k, \cdot)$

Historical background

Robinson (1961): Null electromagnetic fields ~ Shearfree congruences

Sachs equations (1961)
Goldberg-Sachs theorem (1962)
Kerr theorem (1963)

General relativity, exact solutions:
Robinson-Trautman, Kerr, Kerr-Newman, Plebański-Demiański, etc.

Penrose (1967): Twistor geometry

Robinson & Trautman (1985): Cauchy-Riemann structures in optical geometries

Nurowski (1993) Doctoral thesis: Einstein equations and CR geometry

Hill, Lewandowski, Nurowski (2008): Embeddability of CR structures

Baird & Eastwood (2010): CR geometry and conformal foliations

Fino, Leistner, T-C: G-structures (Work in progress)

Almost CR structures

almost CR manifold $(\underline{M}, \underline{H} \subset T\underline{M}, \underline{J} \in \mathrm{End}(\underline{H}))$

$$\dim(\underline{M}) = 2m-1, \quad \mathrm{rk}_{\mathbb{R}}(\underline{H}) = 2m-2, \quad \underline{J}\circ\underline{J} = -\mathrm{Id} \iff \mathbb{C}\otimes\underline{H} = \underline{T}^{1,0} \oplus \overline{\underline{T}^{1,0}},$$

$$\updownarrow$$

$$\underline{\kappa} \in \Omega^1(\underline{M}) \quad \underline{\kappa}|_{\underline{H}} = 0 \qquad d\underline{\kappa}|_{\underline{H}} \rightsquigarrow \text{Levi-form } \underline{\mathcal{L}} : \underline{H}\times\underline{H} \to T\underline{M}/\underline{H}$$

- CR manifold : $T^{1,0}$ involutive $\quad (m=2 \Rightarrow \text{vacuous})$
- contact almost CR : $\underline{\kappa}\wedge(d\underline{\kappa})^{m-1} \neq 0$
- subconformal contact almost CR : $\underline{\kappa}\wedge(d\underline{\kappa})^{m-1} \neq 0$ and $\underline{\mathcal{L}}\circ\underline{J}$ symmetric (conformal structure on $\underline{H}$)

Congruence of null geodesics

$(\mathcal{M}, [g])$ n-dim conformal Lorentzian mfd

null line distribution $K \subset T\mathcal{M}$, $[g]|_K = 0$ $\Longrightarrow$ $K \subset K^\perp$

Screen bundle $K^\perp / K =: H$, rank $n-2$, conformal Riemannian structure $[g|_H]$

(Local) congruence of null curves = flow of non-vanishing $k \in \Gamma(K)$

1-forms: $[\kappa] = [g](k, \cdot) \in \mathrm{Ann}(K^\perp)$

k geodesic $\Longleftrightarrow$ $\mathcal{L}_k [\kappa] = 0$ $\Longrightarrow$ $\mathcal{L}_k [d\kappa] = 0$

Proposition: The screen bundle $H = K^\perp / K$ of a null geodetic congruence on $(\mathcal{M}, [g])$ descends to a distribution $\underline{H}$ of corank 1 on its leaf space $\underline{\mathcal{M}}$

From congruences of null geodesics to almost CR structures

Aim: Given null geodetic congruence $K \subset T\mathcal{M}$ where $n = \dim \mathcal{M} = 2m$
Produce an almost CR structure on leaf space $(\underline{\mathcal{M}}, \underline{H})$

Idea: Use an almost Hermitian structure J on $K^\perp / K = H$

Fact:

$\dim \mathcal{M} = 2m$	4	6	8	10
$O(2m-2)/U(m-1)$	point	$\mathbb{CP}^1 \times \mathbb{CP}^1$	$\mathbb{CP}^3$	$Q^6 \times Q^6 \subset \mathbb{CP}^7 \times \mathbb{CP}^7$

space of all Hermitian structures on H_x $\qquad \dim_{\mathbb{R}} = (m-1)(m-2)$

$$\rightsquigarrow \begin{cases} n = 4, & \text{unique } J \\ n > 4, & \text{no single } J \text{ in general.} \end{cases}$$

Almost Robinson structures

$(\mathcal{M},[g])$ 2m-dim conformal Lorentzian manifold

Almost Robinson structure on $(\mathcal{M},g)$: [Nurowski-Trautman, 2002]

$$N\subset T^{\mathbb{C}}\mathcal{M}, \qquad g^{\mathbb{C}}\big|_N=0, \qquad \mathrm{rk}_{\mathbb{C}} N = m,$$

$$N\cap\overline{N} = K^{\mathbb{C}}, \qquad N+\overline{N} = (K^{\mathbb{C}})^{\perp}$$

Prop $\quad N \iff \left(K,\ J\in \mathrm{End}(K^{\perp}/K) \text{ almost Hermitian}\right)$

$$0 \to K^{\mathbb{C}} \to N \to T^{1,0} \longrightarrow 0 \qquad\qquad T^{1,0}\subset \left(K^{\perp}/K\right)^{\mathbb{C}}$$

Differential conditions:

$[\Gamma(K),\Gamma(N)]\subset\Gamma(N) \iff K$ geodetic and $(\underline{\mathcal{M}},\underline{H},\underline{J})$ almost CR

$[\Gamma(N),\Gamma(N)]\subset\Gamma(N) \iff$ ——— $(\underline{\mathcal{M}},\underline{H},\underline{J})$ CR manifold

Optical invariants of congruences of null geodesics

$(\mathcal{M},[g])$ n-dim conformal Lorentzian mfd w/ congruence of null geodesics.

Two invariants:

- Twist $\tau := d\kappa|_H \quad \rightsquigarrow \quad \tau \in \Lambda^2 H^*$
- Shear $\sigma := \mathcal{L}_k [g_H] \quad \rightsquigarrow \quad \sigma \in \odot^2_\circ H^*$

k geodetic, i.e. $\mathcal{L}_k[\kappa] = 0 \Rightarrow \mathcal{L}_k[d\kappa] = 0 \Longrightarrow \underline{\tau} \in \Lambda^2 \underline{H}^*$

Twist descends to $(\underline{\mathcal{M}}, \underline{H})$

Example: $n = 2m$, rank $\underline{H} = 2m-1$

$\kappa \wedge (d\kappa)^{m-1} \neq 0 \Rightarrow \underline{\kappa} \wedge (d\underline{\kappa})^{m-1} \neq 0 \Longrightarrow (\underline{\mathcal{M}}, \underline{H})$ contact structure

Shearfree congruence of null geodesics

K shearfree geodetic $\iff \mathcal{L}_k [g_H] = 0$

- conformal structure

  $[g_H]$ on $H = K^\perp / K \longrightarrow \mathcal{M} \quad \rightsquigarrow \quad [g_{\underline{H}}]$ on $\underline{H} \longrightarrow \underline{\mathcal{M}}$

- Twist endomorphism

  $F := g_H^{-1} \circ \tau \in \mathrm{End}(H) \quad \rightsquigarrow \quad \underline{F} \in \mathrm{End}(\underline{H})$

- Example : $n = 2m$

  $\underline{\kappa} \wedge \underline{\tau}^{m-1} \neq 0 \implies (\underline{\mathcal{M}}, \underline{H}, T\underline{\mathcal{M}} = [\underline{H}, \underline{H}], [g_{\underline{H}}])$ Contact subconformal

Special case $n=4$

Orientation on H: $\varepsilon \in \Lambda^2 H^*$ $\Longrightarrow$ Orientation on $\underline{H}$: $\underline{\varepsilon} \in \Lambda^2 \underline{H}^*$

k shearfree geodesic $\rightsquigarrow$ Complex structure $\underline{J} = \underline{g}^{-1} \circ \underline{\varepsilon}$ on $\underline{H}$

Proposition [Robinson, Trautman, Penrose, Hill, Nurowski, ...]

Null geodesic congruence is shearfree
$\Longleftrightarrow$ leaf space is CR 3-manifold

Twisting $\rightsquigarrow$ $0 \neq \underline{I} \propto \underline{\varepsilon}$ non-degenerate CR structure (contact subconformal)

Twistfree $\rightsquigarrow$ $0 = \underline{I}$ degenerate CR structure

Integrability condition: $W(k,\cdot,k,\cdot) = 0 \mod \kappa \odot \alpha$

Integrability condition ($n \geqslant 4$)

Proposition

$(\mathcal{M}, [g])$ n-dim conf. Lorentzian mfd.

Let k be the generator of a shearfree congruence of null geodesics with twist τ.

Then the Weyl tensor W satisfies

$$W_{k,\cdot}(k) + \tau \circ \tau + \frac{1}{n-2} \|\tau\|^2 \delta_H = 0 \qquad \text{mod } \kappa \otimes v + \alpha \otimes k$$

(Here $\tau \circ \tau \sim \tau^i{}_k \tau^k{}_j$)

Cf: Ortaggio-Pravda-Pravdova (2007)

Consequences

$$\tau\circ\tau+\frac{1}{n-2}\|\tau\|^2\delta_H=0 \;-(*) \iff W(k,\cdot,k,\cdot)=0 \mod \varkappa\odot\alpha$$

$$(*)\Rightarrow\begin{cases} n \text{ odd} \Rightarrow \tau=0 & \text{OPP (2007)}\\ n \text{ even and } \tau\neq 0 \Longrightarrow & J\circ J=-\delta_H \quad \text{where } J\propto g^{-1}\circ\tau\end{cases}$$

Twist defines an almost Hermitian structure

Theorem

Let $(\mathcal{M},[g])$ be $2m$-dim conformal Lorentzian mfd admitting twisting shearfree null geodesic congruence k

Then the twist defines a subconformal contact almost CR manifold on the leaf space $(\underline{\mathcal{M}},\underline{H})$

$$\iff \quad W(k,\cdot,k,\cdot)=0 \mod \varkappa\odot\alpha$$

Examples

- k null conformal Killing field, i.e. $\mathcal{L}_k[g]=0$ $\Longrightarrow$ Shearfree geodetic

  $\kappa = g(k,\cdot)$, $\quad \tau = d\kappa$, $\quad \Psi = \Box\kappa + \dots$

  Prolongation $\rightsquigarrow$ $W(k,\cdot,\cdot,k) = \tau\circ\tau + \Psi(k)g - 2\kappa\odot\Psi$

- Fefferman conformal structures

  $(\underline{M}, \underline{H}, [g|_{\underline{H}}])$ $\rightsquigarrow$ S^1-bundle $(M,[g])$ + null cvf k

- Sub-conformal contact structure $\rightsquigarrow$ τ "generic" $\quad W(k,\cdot,k,\cdot)\neq 0$
- subconformal contact almost CR structure $\rightsquigarrow$ $\tau \sim J$ $\quad W(k,\cdot,k,\cdot)=0$
- subconformal contact CR structure $\rightsquigarrow$ $\tau\sim J$ integrable $\quad W(k,\cdot,\cdot,\cdot)=0$

Thank you
for
your attention!