Green operators for low regularity spacetimes

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How can we probe the singularity structure of spacetime?

- point particles: Raychaudhuri equation $\rightsquigarrow$ singularity theorems (Hawking, Penrose 70)
- classical test fields: Sobolev regularity $\rightsquigarrow$ strong cosmic censorship (Dafermos 01) , L^2-curvature conjecture (Klainerman, Rodnianski, Szeftel 12)

In this talk we build a case for the following relationship

quantum test fields: Sobolev wave front set $\rightsquigarrow$ physical adiabatic states

Let (M, g) be a smooth Lorentzian manifold. An algebraic quantisation of the wave equation $(\Box_g - m^2)\phi = 0$ is a 'map':

$$\left\{ \begin{array}{l} \text{Classical solutions } \phi \\ \text{(causal propagator)} \end{array} \right\} \to \left\{ \begin{array}{l} C^*\text{-Algebra } \mathcal{A} \\ \text{Positive functionals on } \mathcal{A} \end{array} \right\}$$

Construction of $\mathcal{A}$:

$$\phi \to (\phi, \Omega) \to W(\phi) * W(\tilde{\phi}) := e^{\frac{i}{2}\Omega(\phi,\tilde{\phi})} W(\phi + \tilde{\phi})$$

The algebra $\mathcal{A}$ is the algebra of quantum observables.
The quantum states, ω, are given by positive linear functional on the algebra $(\omega(a^* a) > 0 \quad \forall a \in \mathcal{A}, \quad \omega(I) = 1)$.
Quantum test field= Fock representation (GNS construction)

$$\omega(a) = \langle \Psi | \pi(a) | \Psi \rangle$$

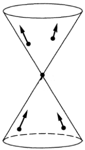
Requirement: energy momentum tensor T_{ab} has to be an observable. "$\langle\Psi|\hat{T}_{ab}|\Psi\rangle$" (Kay, Wald 91; Radzikowski 92).
The two-point function of a state ω is given by

$$\omega_2(\phi, \tilde{\phi}) =: -\frac{\partial^2}{\partial s \partial t}\{\omega(W(s\phi + t\tilde{\phi})e^{ist\Omega(\phi,\tilde{\phi})})\}|_{s=t=0}$$

Definition

A state ω_H is a **Hadamard state** if its two point function ω_{2H} is a distribution $D'(M \times M)$ and satisfies the following wavefront set condition

$$WF(\omega_{2H}) = C^+$$



There is a generalisation of Hadamard states called adiabatic states (Junker, Schrohe 02).

Definition

A state ω_N is an **adiabatic state** of order $N \in \mathbb{R}$ if its two-point function ω_{2N} satisfies the following H^s-wavefront set condition for all $s \leq N + \frac{3}{2}$

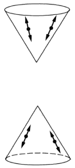
$$WF^{s}(\omega_{2N}) \subset C^{+}$$

$(x, \chi) \notin WF^s(u)$ if at x there exists a conic neighbourhood Γ of χ and $\varphi \in C_0^\infty, \varphi(x) \neq 0$ such that

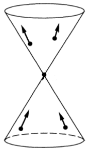
$$\int_\Gamma (1 + |\xi|^2)^s |\widehat{\varphi u}(\xi)|^2 d^n\xi < \infty.$$

In this talk a Green operator is

- **Causal propagator**



- **Two-point function** of quasi-free Hadamard (adiabatic) states



Low regularity Lorentzian geometry

- **physical models**: general relativistic fluids (LeFloch, Xiang 16), impulsive gravitational waves (Luk, Rodnianski 15), cosmic strings (Sperhake, Sjodin, Vickers 00)

$$\Updownarrow$$

- **mathematical foundations**: causal structure (Chruściel, Grant 12), global hyperbolicity (Bernal, Sanchez 07; Sämann 16), singularity theorems (Kunzinger, Steinbauer, Vickers 15; Graf, Grant, Kunzinger, Steinbauer 17)

Causal propagator Joint work with James Vickers.
$g_{ab} \in C^1([0,T], L^\infty(\Sigma))$, $g_{\alpha\beta}(t_0,\cdot)$ uniformly elliptic.
Kernel representation of causal propagators.

- Introduce orthogonal basis $\{w_k\}_{k=1}^\infty$ in $H^1(\Sigma)$
- Insert approximate solution $\phi^{(m)}(t,x) = \sum_{k=1}^m d_k^{(m)}(t) w_k(x)$ into wave equation. This gives a first order system of ODEs for $d_k^{(m)}(t)$.
- Can find a Green's matrix $G_{jk}(s,t)$ for the system of ODEs
- Use this to construct approximate Green's function $G^{(m)}(x,y:s,t)$ given by

$$G^m(x,y:s,t) = \sum_{k,j=1}^m G_{kj}(s,t) w_k(x) w_j^*(y)$$

- G^m converges as $m \to \infty$ to a unique distributional Green's function.
  [Vickers,-; Journal of Physics: Conference Series, 968(1), 1-14] See also (Dereziński,Siemssen 18) for another approach.

Adiabatic ground states Joint work with Elmar Schrohe.
$g_{ab} \in C^{1+\epsilon}$, g_{ab} ultrastatic.
Kernel analysis of ground states ω_g.

- Construct ground ω_g (use essential self-adjointness of $(\Delta_{LB} - m^2)$)
- Notice that the two point function of ω_g restricted to a hypersurface Σ is given by $F(\cdot,\cdot) = (\cdot, (\Delta_{LB} - m^2)^{-\frac{1}{2}} \cdot)_{L^2(\Sigma)}$.
- Define the non-smooth PDO $A^{-\frac{1}{2}} = \int_\Gamma \frac{\lambda^{-\frac{1}{2}} d\lambda}{h^{ij}\xi_i\xi_j - \lambda}$ where h^{ij} is the inverse of the induce Riemannian metric on Σ.
- Can obtain estimates $||(\Delta_{LB} - m^2)^{-\frac{1}{2}} - A^{-\frac{1}{2}}||_{p,q} < \infty$
- Use mapping properties to analyse kernel. For example, $p = 0, q > \frac{n}{2}$ kernel is L^2

Summary and future work:

- The Sobolev wave front set of Green operators depend on the regularity of the spacetime.
- Adiabatic states are the natural candidates for grounds states in non-smooth spacetimes.
- Adiabatic states of certain order are physically reasonable (microlocal-approach to probe spacetime singularities).
- Adiabatic states are physically discernible (experimental possibilities?).

Thank you.