# “Timelike surfaces in Minkowski space with a canonical null direction”

Gabriel Ruiz Hernández

www.matem.unam.mx/gruiz

Instituto de Matemáticas, UNAM

IX International Meeting on Lorentzian Geometry

Banach Center

June 22, 2018. Warsaw Poland

# Minkowski space

BASED ON A JOINT WORK WITH: **Victor Hugo Patty from Bolivia.**

We consider $\mathbb{R}^{n,1}$ the $(n+1)-$dimensional Minkowski space defined by $\mathbb{R}^{n+1}$ endowed with the Lorentz metric

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + \ldots + dx_{n+1}^2.$$

An immersed surface $M^2 \subset \mathbb{R}^{n,1}$ is timelike

- if the metric $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^{n,1}$ induces a Lorentzian metric on $M$.
- (equivalently) for every $p \in M$, $T_pM$ is a timelike plane.
- (equivalently) for every $p \in M$, the two codimensional subspace $(T_pM)^{\perp}$ is spacelike.
- (equivalently) for every $p \in M$, $T_pM$ is a plane that contains a timelike line.

# Canonical null direction

**Definition**

*We say that a timelike surface $M$ immersed in $\mathbb{R}^{n,1}$ has a* **canonical null direction** *with respect to $Z$, if the tangent part $Z^{\top}$ of $Z$ is a lightlike vector field along $M$, i.e. $Z^{\top}$ is nowhere zero and $\langle Z^{\top}, Z^{\top}\rangle = 0$. Here $Z$ is a constant vector field in $\mathbb{R}^{n,1}$ with $|Z| := |\langle Z, Z\rangle|^{1/2} = 1$. We will say that $Z$ induces a null direction on the surface.*

We have the decomposition:

$$Z = Z^{\top} + Z^{\perp} \in TM + (TM)^{\perp}.$$

So,

$$\langle Z, Z\rangle = \langle Z^{\top}, Z^{\top}\rangle + \langle Z^{\perp}, Z^{\perp}\rangle = \langle Z^{\perp}, Z^{\perp}\rangle$$

and therefore $Z$ is necessarily a space like vector:

$$\langle Z, Z\rangle = 1.$$

# Notions on Geometry of Submanifolds

Let $M \subset \mathbb{R}^{n,1}$ be an timelike immersed surface with the induced metric. Let $D$ (resp. $\nabla$) be the Levi-Civita connection of $\mathbb{R}^{n,1}$ (resp. $M$), $X$, $Y$ vector fields on $M$ and $\xi, \nu$ sections of $(TM)^{\perp}$.

- Shape Operator and Normal Connection:

$$A_{\xi}(X) := -(D_X \xi)^{\top}, \text{ and } \nabla^{\perp}_X \xi := (D_X \xi)^{\perp}.$$

- Second Fundamental Form:

$$II : TM \times TM \to (TM)^{\perp} \text{ is } II(X, Y) := D_X Y - \nabla_X Y.$$

## Lemma

*In any codimension:*
*Since $Z$ is a parallel vector field and $Z^\top$ is ligthlike on $M$*

$$A_{Z^\perp}(Z^\top) = 0 \qquad \nabla_{Z^\top} Z^\top = 0,\ \langle II(Z^\top, Z^\top), Z^\perp\rangle = 0.$$

*The integral curves of $Z^\top$ are light like geodesics of $M$.*

PROOF:

$$Z \text{ is parallel: } \nabla_X Z^\top = A_{Z^\perp}(X) \text{ and } \nabla^\perp_X Z^\perp = -II(Z^\top, X), \quad (1)$$

for all $X \in TM$.

$$\begin{aligned} \langle A_{Z^\perp}(Z^\top), X\rangle &= \langle II(Z^\top, X), Z^\perp\rangle = -\langle \nabla^\perp_X Z^\perp, Z^\perp\rangle \\ &= -\tfrac{1}{2} X \cdot \langle Z^\perp, Z^\perp\rangle = -\tfrac{1}{2} X \cdot \langle Z, Z\rangle = 0. \\ \langle II(Z^\top, Z^\top), Z^\perp\rangle &= -\langle \nabla^\perp_{Z^\top} Z^\perp, Z^\perp\rangle = 0. \end{aligned}$$

**Proposition**

*If M is an immersed surface in $\mathbb{R}^{2,1}$ with a canonical null direction if and only if M is minimal and flat.*

PROOF:
In this case $-II(Z^\top, X) = \nabla^\perp_X Z^\perp = 0$. By the Gauss Equation:

$$\langle R^\nabla(X, Z^\top)X, Z^\top\rangle = \langle II(X, X), II(Z^\top, Z^\top)\rangle - \langle II(X, Z^\top), II(X, Z^\top)\rangle.$$

Then $K = 0$.
If $W$ completes $Z^\top$ into a null frame with $\langle Z^\top, W\rangle = -1$: The mean curvature vector on this basis with lightlike vectors

$$\vec{H} := -II(Z^\top, W) = 0.$$

## Theorem

*A timelike surface M in $\mathbb{R}^{2,1}$ with a canonical null direction Z can be locally parametrized by*

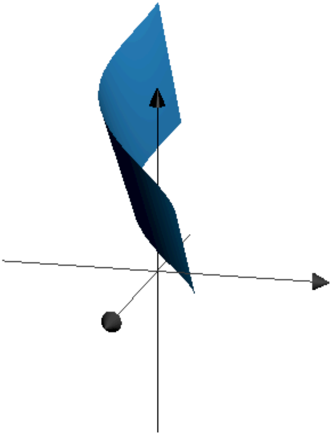
$$\psi(x, y) = \alpha(x) + y\ T_0, \tag{2}$$

*where $\alpha(x)$ is a light-like curve in $\mathbb{R}^{2,1}$, $T_0$ is some constant lightlike vector and the vectors $\alpha'(x)$ and $T_0$ are linearly independent for every $x$.*

Remark: Any timelike minimal surface in $\mathbb{R}^{2,1}$ can be parametrized as: $\psi(x, y) = \alpha(x) + \beta(y)$, with $\alpha$, $\beta$ lightlike curves.

# A surface in $\mathbb{R}^{2,1}$ with a canonical null direction

Figure: $\alpha(x) = (\sin(x), \cos(x), x), \ T_0 = (1, 0, 1)$



We complete $Z^\top$ into a pseudo orthonormal frame on $M$:$\{Z^\top, W, \}$

- $W$ a lightlike vector field tangent to $M$, $\langle W, W\rangle = 0$,
- $\langle Z^\top, W\rangle = -1$.

In this frame

$$\vec{H} := \frac{1}{2}\mathrm{tr}_{\langle,\rangle} II = -II(Z^\top, W).$$

## Proposition

$$|\vec{H}|^2 = -\langle \nabla^\perp_W \vec{H}, Z^\perp\rangle.$$

*In particular, if the mean curvature vector $\vec{H}$ is parallel then the surface $M$ is minimal, i.e. $\vec{H} = 0$.*

# Gaussian and Normal Curvatures

We define the function $a := \langle II(W, W), Z^{\perp} \rangle$
Normal Curvature when $n = 2$: $K_N := \frac{\langle R^{\perp}(X,Y)\xi,\nu \rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$.

## Corollary

*Let us assume that* $|II(Z^{\top}, Z^{\top})| \neq 0$ *everywhere. Then*

$$K = \frac{\langle R(Z^{\top}, W)Z^{\top}, W\rangle}{|Z^{\top}|^2|W|^2 - \langle Z^{\top}, W\rangle^2} = Z^{\top} \cdot a,$$

*and*

$$K_N = a|II(Z^{\top}, Z^{\top})|.$$

*In particular, if M has constant zero normal curvature then the Gauss curvature K is also constant zero.*

Let $f, g : U \subset \mathbb{R}^2 \to \mathbb{R}$ be two smooth functions. Let

$$M := \left\{ (f(x,y), g(x,y), x, y) \in \mathbb{R}^{3,1} \mid x, y \in U \right\} \subset \mathbb{R}^{3,1}$$

be a timelike surface given as a graph (image) of the function $\psi(x,y) = (f(x,y), g(x,y), x, y)$.
We define $F := \langle \psi_x, \psi_y \rangle = -f_x f_y + g_x g_y$.

## Proposition

*Let $M$ be a timelike surface given as a graph in $\mathbb{R}^{3,1}$. Then $M$ has a canonical null direction with respect to $e_4$ (resp. $e_3$) if and only if $\psi_x$ (resp. $\psi_y$) is a lightlike vector field along $M$. In that situation we have*

$$e_4^\top = \frac{1}{F}\psi_x \qquad \left( resp. \;\; e_3^\top = \frac{1}{F}\psi_y \right).$$

$$\vec{H} = \psi_{xy}/F, \;\; K = -\frac{1}{F}\left(\frac{F_x}{F}\right)_y.$$

## Example

Our following surface is a graph with two canonical null directions:

$$\psi(x, y) = (\sinh(x + y), \cosh(x + y), x, y).$$

The induced tangent vectors are
$\psi_x = (\cosh(x + y), \sinh(x + y), 1, 0)$ and
$\psi_y = (\cosh(x + y), \sinh(x + y), 0, 1)$ which are lightlike vector fields. Our surface has two canonical null directions with respect to both $e_3$ and $e_4$. This surface is not minimal because $\psi_{xy} = (\sinh(x + y), \cosh(x + y), 0, 0) \neq 0$. This surface is also flat because $F = \langle \psi_x, \psi_y \rangle = -1$.

## Proposition

*Let M be a minimal timelike surface in $\mathbb{R}^{3,1}$ not contained in a hyperplane. Then M has a canonical null direction with respect to $e_4$ if and only if M can be locally parametrized by*

$$\psi(x, y) = \alpha(x) + \beta(y) \tag{3}$$

*where $\alpha$ and $\beta$ are two lightlike curves and either $\alpha$ or $\beta$ is contained in a timelike hyperplane orthogonal to $e_4$. A surface with a parametrization like this is called a translation surface.*

## Proposition

*Let $M_0$ be a Lorentzian surface in $\mathbb{R}^{2,1}$, $f : M_0 \to \mathbb{R}$ be a given smooth function. Let us consider the surface obtained as the graph of $f$, i.e.*

$$M := \{(p, f(p)) |\; p \in M_0\} \;\subset\; \mathbb{R}^{3,1},$$

*with the induced metric. Then $M$ has a canonical null direction with respect to $e_4$ if and only if $\nabla f$ is a lightlike vector field on $M_0$.*

## Corollary

*Let M be an immersed surface in $\mathbb{R}^{3,1}$. The surface M has a canonical null direction with respect to $e_4$ if and only if locally it can be parametrized as in Proposition above.*

**Proposition**

*Let $M$ be a Lorentzian surface, $f : M \to \mathbb{R}$ be a given smooth function. If the gradient $\nabla f$ is a lightlike vector field then the integral curves of $\nabla f$ are geodesics and $f$ is a harmonic function, i.e. $\triangle f = 0$.*

## Example

Let us consider the timelike surface

$$M_0 := \{(x, \cos(y), \sin(y)) \mid\ x \in \mathbb{R},\ y \in (0, 2\pi)\} \ \subset\ \mathbb{R}^{2,1},$$

and
the function $f : M_0 \to \mathbb{R}$ given by $f(x, \cos(y), \sin(y)) = y - x$.
The level curve $\gamma(y) = (y - c, \cos(y), \sin(y))$ is a lightlike geodesic in $M_0$ for all constant $c \in \mathbb{R}$.
Remark: $\partial_x = e_1$ and $\partial_y = -\sin y\ e_2 + \cos y\ e_3$ are an orthonormal frame along $M_0$.
Indeed, we only have to remark that
$\gamma'(y) = (1, -\sin(y), \cos(y)) = \partial_x + \partial_y$ is a lightlike vector field.
$\nabla f = -(\partial_x f)\partial_x + (\partial_y f)\partial_y = \partial_x + \partial_y = \gamma'(y)$, which is a lightlike vector field on $M_0$;

# Bibliografía

1. V. H. Patty Yujra, G. Ruiz-Hernández.
Timelike surfaces in Minkowski space with a canonical null direction.
Journal of Geometry 2018.

Thank you for your attention!