Extending Calabi's duality

José Antonio Sánchez Pelegrín

Departamento de Geometría y Topología, Universidad de Granada

IX International Meeting on Lorentzian Geometry

Supported by Spanish MINECO and EDRF project MTM2013-47828-C2-1-P.

Contents

In this conference we will show a correspondence between solutions of two second order nonlinear elliptic PDE as well as some geometric consequences. Namely, the minimal and maximal surface equation in the following ambiences:

1. $\mathbb{R}^3$ and $\mathbb{L}^3$
2. Riemannian and Lorentzian warped product spaces

Calabi's duality

Our first result is due to Calabi, who found a correspondence between local solutions to the minimal surface equation in $\mathbb{R}^3$ and local solutions to the maximal surface equation in $\mathbb{L}^3$.

Calabi's duality

Minimal surface

A surface $M \subset \mathbb{R}^3$ is minimal if its mean curvature is identically zero.

Minimal surfaces are critical points for the area functional for all compactly supported normal variations.

Calabi's duality

Minimal surface equation in $\mathbb{R}^3$

Given a smooth function u defined on a domain $\Omega \subset \mathbb{R}^2$, its graph

$$\Sigma_u = \{(p, u(p)) : p \in \mathbb{R}^2\}$$

defines a minimal surface in $\mathbb{R}^3$ if and only if u satisfies

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$$

Calabi's duality

Spacelike surface

An immersed surface M in the Lorentz-Minkowski spacetime $\mathbb{L}^3$ is called spacelike if the induced metric on M is Riemannian.

Calabi's duality

Spacelike surface

An immersed surface M in the Lorentz-Minkowski spacetime $\mathbb{L}^3$ is called spacelike if the induced metric on M is Riemannian.

Maximal surface

A spacelike surface $M \subset \mathbb{L}^3$ is maximal if its mean curvature is identically zero.

Calabi's duality

Maximal surface equation in $\mathbb{L}^3$

Given a smooth function ω defined on a domain $\Omega \subset \mathbb{R}^2$, its graph

$$\Sigma_\omega = \{(p, \omega(p)) : p \in \mathbb{R}^2\}$$

defines a maximal surface in $\mathbb{L}^3$ if and only if ω satisfies

$$\mathrm{div}\left(\frac{D\omega}{\sqrt{1-|D\omega|^2}}\right) = 0,$$

$$|Du|^2 < 1.$$

Calabi's duality

In 1970, Calabi [1] proved that there exists a one-to-one correspondence between solutions to the minimal surface equation in $\mathbb{R}^3$ and solutions to the maximal surface equation in $\mathbb{L}^3$.

[1] E. Calabi, Examples of Bernstein problems for some nonlinear equations, *P. Symp. Pure Math.*, **15** (1970), 223–230.

Calabi's duality

In 1970, Calabi [1] proved that there exists a one-to-one correspondence between solutions to the minimal surface equation in $\mathbb{R}^3$ and solutions to the maximal surface equation in $\mathbb{L}^3$.

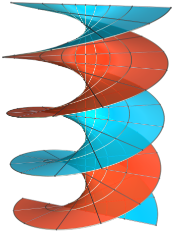
In particular, given a minimal graph Σ_u over a simply connected domain $\Omega \subseteq \mathbb{R}^2$, we can find a maximal graph Σ_ω over Ω and vice versa

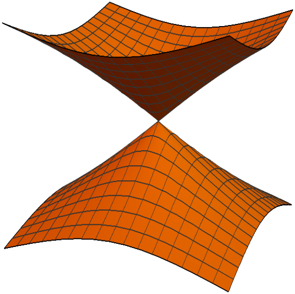
[1] E. Calabi, Examples of Bernstein problems for some nonlinear equations, *P. Symp. Pure Math.*, **15** (1970), 223–230.

Calabi's duality

Examples of dual graphs

This duality converts the helicoid in $\mathbb{R}^3$, $u = \arctan\left(\frac{y}{x}\right)$ to the Lorentzian catenoid $\omega = \sinh^{-1}\left(\sqrt{x^2+y^2}\right)$.

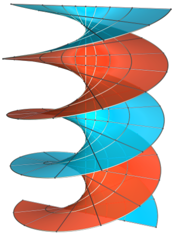


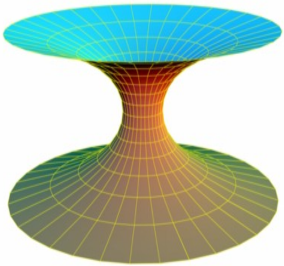


Calabi's duality

Examples of dual graphs

Conversely, the helicoid in $\mathbb{L}^3$, $\omega = \arctan\left(\frac{y}{x}\right)$ recovers the Euclidean catenoid $u = \cosh^{-1}\left(\sqrt{x^2+y^2}\right)$.





Calabi's duality

Bernstein Theorem [2]

The only entire solutions to the minimal surface equation in $\mathbb{R}^3$ are the affine functions.

---

[2] S. Bernstein, Sur un théorème de géométrie et son aplication aux équations aux dérivées partielles du type elliptique, *Comm. de la Soc. Math. de Kharkow*, **15** (1915), 38–45.

[1] E. Calabi, Examples of Bernstein problems for some nonlinear equations, *P. Symp. Pure Math.*, **15** (1970), 223–230.

Calabi's duality

Bernstein Theorem [2]

The only entire solutions to the minimal surface equation in $\mathbb{R}^3$ are the affine functions.

Calabi-Bernstein Theorem [1]

The only entire solutions to the maximal surface equation in $\mathbb{L}^3$ are the affine functions satisfying $|Du| < 1$.

[2] S. Bernstein, Sur un théorème de géométrie et son aplication aux équations aux dérivées partielles du type elliptique, *Comm. de la Soc. Math. de Kharkow*, **15** (1915), 38–45.

[1] E. Calabi, Examples of Bernstein problems for some nonlinear equations, *P. Symp. Pure Math.*, **15** (1970), 223–230.

Calabi's duality

This correspondence was also obtained using the local Enneper-Weierstrass representation of a minimal surface in $\mathbb{R}^3$ and the local Enneper-Weierstrass representation of a maximal surface in $\mathbb{L}^3$, reproving with a very different approach the equivalence between Bernstein and Calabi-Bernstein theorems [3].

[3] F.J.M. Estudillo and A. Romero, Generalized maximal surfaces in Lorentz-Minkowski space $\mathbb{L}^3$, *Math. Proc. Cambridge*, **111** (1992), 515–524.

Calabi's duality

Later, this duality was extended to product spaces [4]. Hence, it was found a local correspondence between solutions of the minimal surface equation in a 3-dimensional Riemannian product space $M^2 \times \mathbb{R}$ and solutions of the maximal surface equation in a 3-dimensional Lorentzian product space $M^2 \times (-\mathbb{R})$.

[4] A.L. Albujer and L.J. Alías, Calabi-Bernstein results for maximal surfaces in Lorentzian product spaces, *J. Geom. Phys.*, **59** (2009), 620–631.

Calabi's duality

Later, this duality was extended to product spaces [4]. Hence, it was found a local correspondence between solutions of the minimal surface equation in a 3-dimensional Riemannian product space $M^2 \times \mathbb{R}$ and solutions of the maximal surface equation in a 3-dimensional Lorentzian product space $M^2 \times (-\mathbb{R})$.

This correspondence was used to obtain non-trivial examples of entire maximal graphs in the Lorentzian product space $\mathbb{H}^2 \times (-\mathbb{R})$.

[4] A.L. Albujer and L.J. Alías, Calabi-Bernstein results for maximal surfaces in Lorentzian product spaces, *J. Geom. Phys.*, **59** (2009), 620–631.

Duality in warped product spaces

Recently, we have been able to extend this duality to a wider class of ambient manifolds. Namely, we have found a correspondence between local solutions of the minimal surface equation in a 3-dimensional Riemannian warped product space and local solutions of the maximal surface equation in a related 3-dimensional standard static spacetime.

Duality in warped product spaces

Riemannian warped product space

The Riemannian warped product $B \times_{\frac{1}{\sqrt{\gamma}}} \mathbb{R}$ will denote the product manifold $B \times \mathbb{R}$, being (B, g_B) a connected Riemannian surface, endowed with the Riemannian metric

$$\overline{g} = \pi_B^*(g_B) + \frac{1}{\gamma(\pi_B)} \pi_{\mathbb{R}}^*(dt^2),$$

where π_B and $\pi_{\mathbb{R}}$ denote, respectively, the projections on B and $\mathbb{R}$ and γ is a smooth positive function on B.

Duality in warped product spaces

Minimal surface equation in $B \times_{\sqrt{\frac{1}{\gamma}}} \mathbb{R}$

The graph of a smooth function u defined on a domain $\Omega \subset B$,

$$\Sigma_u = \{(p, u(p)) : p \in B\}$$

defines a minimal surface in $B \times_{\sqrt{\frac{1}{\gamma}}} \mathbb{R}$ if and only if u satisfies

$$\mathrm{div}\left(\frac{Du}{\sqrt{\gamma + |Du|^2}}\right) = \frac{1}{2\gamma}\frac{g_B(D\gamma, Du)}{\sqrt{\gamma + |Du|^2}}. \tag{R}$$

Duality in warped product spaces

Standard static spacetimes

A 3-dimensional standard static spacetime $B \times_{\sqrt{\gamma}} (-\mathbb{R})$ is defined as the product manifold $B \times \mathbb{R}$, being (B, g_B) a connected Riemannian surface, endowed with the Lorentzian metric

$$\overline{g} = \pi_B^*(g_B) - \gamma(\pi_B)\pi_{\mathbb{R}}^*(dt^2),$$

where π_B and $\pi_{\mathbb{R}}$ denote, respectively, the projections on B and $\mathbb{R}$ and γ is a smooth positive function on B.

Duality in warped product spaces

Physical relevance of standard static spacetimes

In these spacetimes there exists an irrotational timelike Killing vector field that defines a family of observers that measure a spatial universe that does not change with time.

Standard static spacetimes include some relevant models such as Einstein static universe and (exterior) Schwarzschild spacetime.

Duality in warped product spaces

Maximal surface equation in $B \times_{\sqrt{\gamma}} (-\mathbb{R})$

Given a smooth function ω defined on a domain $\Omega \subset B$,

$$\Sigma_\omega = \{(p, \omega(p)) : p \in B\}$$

defines a maximal surface in $B \times_{\sqrt{\gamma}} (-\mathbb{R})$ if and only if ω satisfies

$$\operatorname{div}\left(\frac{D\omega}{\sqrt{\frac{1}{\gamma} - |D\omega|^2}}\right) = \frac{\gamma}{2} \frac{g_B\left(D\left(\frac{1}{\gamma}\right), D\omega\right)}{\sqrt{\frac{1}{\gamma} - |D\omega|^2}}, \tag{L.1}$$

$$|D\omega|^2 < \frac{1}{\gamma}. \tag{L.2}$$

Duality in warped product spaces

Theorem [5]

Let $\Omega \subset B$ be a simply connected domain. Then, there exists a non-trivial (i.e., non-constant) solution of the minimal surface equation in $B \times_{\frac{1}{\sqrt{\gamma}}} \mathbb{R}$ on Ω if and only if there exists a non-trivial solution of the maximal surface equation in $B \times_{\sqrt{\gamma}} (-\mathbb{R})$ on Ω.

[5] J.A.S. Pelegrín, A. Romero and R.M. Rubio, An extension of Calabi's correspondence between the solutions of two Bernstein problems to more general elliptic nonlinear equations, *Math. Notes*, **5** (2018), to appear.

Duality in warped product spaces

Sketch of the proof

1. If we consider on B the conformal metric

$$\widetilde{g} = \frac{1}{\sqrt{\gamma}}\, g_B,$$

we can rewrite (R) as follows

$$\widetilde{\operatorname{div}}\left(\frac{Du}{\sqrt{\gamma + |Du|^2}}\right) = 0. \qquad (\widetilde{\mathrm{R}})$$

Sketch of the proof

2. Analogously, with the conformal change of metric

$$\widehat{g} = \sqrt{\gamma}\, g_B,$$

we can write (L) as

$$\widehat{\operatorname{div}}\left(\frac{D\omega}{\sqrt{\frac{1}{\gamma} - |D\omega|^2}}\right) = 0, \qquad (\widehat{\mathrm{L}}.1)$$

$$|D\omega|^2 < \frac{1}{\gamma}. \qquad (\widehat{\mathrm{L}}.2)$$

Duality in warped product spaces

Sketch of the proof

3. If we assume now the existence of a non-trivial solution u of $(\widetilde{\mathrm{R}})$ on $\Omega \subset B$, we have

$$d * \left(U^{\widetilde{\flat}}\right) = 0,$$

where $U := \dfrac{Du}{\sqrt{\gamma + |Du|^2}}$.

Duality in warped product spaces

Sketch of the proof

3. If we assume now the existence of a non-trivial solution u of $(\widetilde{\mathrm{R}})$ on $\Omega \subset B$, we have

$$d * \left(U^{\widetilde{\flat}}\right) = 0,$$

where $U := \dfrac{Du}{\sqrt{\gamma + |Du|^2}}$.

4. Since Ω is simply connected, classical Poincaré's lemma ensures the existence of $\omega \in C^\infty(B)$ such that

$$d\,\omega = *\left(U^{\widetilde{\flat}}\right).$$

Duality in warped product spaces

Sketch of the proof

5. After several computations we obtain that ω satisfies

$$* \left(\left(\frac{D\omega}{\sqrt{\frac{1}{\gamma} - |D\omega|^2}} \right)^{\widehat{\flat}} \right) = d(-u)$$

and

$$|D\omega|^2 < \frac{1}{\gamma}.$$

Hence, ω is a non-trivial solution of $(\widehat{\mathrm{L}})$ on Ω. The converse is proved in the same way.

Duality in warped product spaces

As a consequence of this correspondence, we can obtain new Calabi-Bernstein type results from known Bernstein type results. For instance, knowing

Theorem [6]

The only entire solutions of equation (R) on the 2-dimensional sphere $\mathbb{S}^2$ endowed with a Riemannian metric are the constants.

---

[6] A. Romero and R.M. Rubio, Bernstein-type Theorems in a Riemannian Manifold with an Irrotational Killing Vector Field, *Mediterr. J. Math.*, **13** (2016), 1285–1290.

Duality in warped product spaces

Combining the previous result with our duality theorem in these ambiences we obtain

Corollary [5]

The only entire solutions of equation (L) on the 2-dimensional sphere $\mathbb{S}^2$ endowed with a Riemannian metric are the constants.

[5] J.A.S. Pelegrín, A. Romero and R.M. Rubio, An extension of Calabi's correspondence between the solutions of two Bernstein problems to more general elliptic nonlinear equations, *Math. Notes*, **5** (2018), to appear.

Thank you

for your attention!