Lorentz manifolds with restricted conformal group of maximal dimension

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Outline

Motivation
The Riemannian Case
The Lorentzian Case

The Model Spaces
The Einstein Static Universe $\mathcal{E}^{1,n}$
Compact Quotients of $\mathcal{E}^{1,n}$: The Integral Forms $\mathcal{E}^{1,n}_{\mathrm{I},k}$ and $\mathcal{E}^{1,n}_{\mathrm{II},k}$

Main Results
The Canonical Covering of $\mathrm{O}^{\uparrow}_{+}(2, n+1)$
The Restricted Conformal Group of $\mathcal{E}^{1,n}$, $\mathcal{E}^{1,n}_{\mathrm{I},k}$, and $\mathcal{E}^{1,n}_{\mathrm{II},k}$
Lorentz Manifolds with Large Restricted Conformal Group

Outline

The Riemannian Case

- An n-dimensional Riemmannian manifold $(\mathbb{M}, g)$, $n \geq 3$, with a conformal group of maximal dimension is conformally diffeomorphic to $\mathbb{S}^n$ with its natural conformal structure.
- Any Riemannian $(\mathbb{M}, g)$ with an essential group of conformal transformations (i.e., strictly larger than the isometry group of any metric in the conformal class of g) is conformally equivalent to $\mathbb{S}^n$ or to $\mathbb{E}^n \cong \mathbb{S}^n \setminus \{\infty\}$.
- This is the Lichnerowicz conjecture proved by Lelong-Ferrand and Obata (compact case) and Alekseevsky (general case).

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The Lorentzian Case

- The result of Lelong-Ferrand, Obata and Alekseevsky fails in the non-Riemannian (in particular Lorentzian) case.
- There are many examples of Lorentz manifolds with an essential conformal group (Frances, 2002).
- There are many examples of nonequivalent conformal Lorentz manifolds with conformal group of maximal dimension.

Examples

- The Einstein universes $\mathcal{E}_{\mathrm{I}}^{1,n} \cong \mathbb{S}^1 \times \mathbb{S}^n$ and $\mathcal{E}_{\mathrm{II}}^{1,n} \cong \mathbb{S}^1 \times \mathbb{S}^n/\mathbb{Z}_2$, the conformal compactifications of Minkowski $(n+1)$-space.
- The Einstein static universe $\mathcal{E}^{1,n} \cong \mathbb{R} \times \mathbb{S}^n$, with conformal structure induced by the product metric $-dt^2 + g_{\mathbb{S}^n}$; it is the universal covering of $\mathcal{E}_{\mathrm{I}}^{1,n}$ and $\mathcal{E}_{\mathrm{II}}^{1,n}$.

Posing the Problem

- Let $\mathbb{M}$ be an oriented, time-oriented, conformal Lorentz manifold of dimension $n+1 \geq 3$.
- Let $\mathcal{C}_+^\uparrow(\mathbb{M})$ be the group of conformal transformations of $\mathbb{M}$ that preserve the orientation and the time-orientation (the restricted conformal group).
- It is known that $\mathcal{C}_+^\uparrow(\mathbb{M})$ has dimension $\leq \binom{n+3}{2}$.

Problem

Determine the conformal Lorentz manifolds $\mathbb{M}^{n+1}$, $n \geq 2$, whose restricted conformal group $\mathcal{C}_+^\uparrow(\mathbb{M})$ has dimension equal to $\binom{n+3}{2}$.

Examples

1. $\mathcal{C}_+^\uparrow(\mathcal{E}_{\mathrm{I}}^{1,n}) \cong \mathrm{O}_+^\uparrow(2, n+1)$, identity component of $\mathrm{O}(2, n+1)$.
2. $\mathcal{C}_+^\uparrow(\mathcal{E}_{\mathrm{II}}^{1,n})$ (n odd) $\cong \mathrm{O}_+^\uparrow(2, n+1)/\{\pm I\}$.
3. $\mathcal{C}_+^\uparrow(\mathcal{E}^{1,n}) \cong \widehat{\mathrm{O}}_+^\uparrow(2, n+1)$ (canonical covering of $\mathrm{O}_+^\uparrow(2, n+1)$).

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The Restricted Conformal Group of a Lorentz Manifold

- Let $\mathbb{M}^{n+1}$ be an oriented, time-oriented, conformal Lorentz manifold ($n+1 \geq 3$).
- The set $\mathrm{CO}_+^\uparrow(\mathbb{M})$ of all positive linear conformal frames of $\mathbb{M}$ defines a principal fiber bundle $\pi_\mathrm{C} : \mathrm{CO}_+^\uparrow(\mathbb{M}) \to \mathbb{M}$, with structure group $\mathrm{CO}_+^\uparrow(1,n) = \left\{ r\mathrm{B} \mid r > 0,\, \mathrm{B} \in \mathrm{O}_+^\uparrow(1,n) \right\}$.
- As a particular instance of the procedure of prolongation of a G-structure, from $\mathrm{CO}_+^\uparrow(\mathbb{M})$ one can construct the Cartan conformal bundle $\mathrm{Q}(\mathbb{M})$.
- $\mathrm{Q}(\mathbb{M})$ admits an absolute parallelism, the normal parallelism (normal Cartan connection).

The Restricted Conformal Group of a Lorentz Manifold

- By classical results of E. Cartan and S. Kobayashi on the transformation group of a manifold with an absolute parallelism, we have the following.

Theorem (The group of RCTs of a Lorentz manifold)

- *Let* G *be a connected Lie group acting effectively on* $\mathbb{M}$ *by restricted conformal transformations (RCTs).*
- *For any* $\dot{\mathcal{A}}_* \in \mathrm{Q}(\mathbb{M})$*, the map* $\mathrm{j} : \mathrm{G} \to \mathrm{Q}(\mathbb{M})$*,* $g \mapsto \dot{g}(\dot{\mathcal{A}}_*)$ *is an embedding of* G *onto a closed submanifold of* $\mathrm{Q}(\mathbb{M})$*.*
- *In particular,* $\dim \mathrm{G} \leq \dim \mathrm{Q}(\mathbb{M}) = \binom{n+3}{2}$*.*
- *If* $\dim \mathrm{G} = \binom{n+3}{2}$*,* $\mathrm{Q}(\mathbb{M})$ *has a Lie group structure with neutral element* $\dot{\mathcal{A}}_*$*.*

Outline

The Einstein Static Universe

- The Einstein static universe $\mathcal{E}^{1,n}$ is the hypersurface of $\mathbb{R} \times \mathbb{R}^{n+1}$ defined by

$$\mathcal{E}^{1,n} := \left\{(\tau, x) \in \mathbb{R} \times \mathbb{R}^{n+1} \mid {}^t x x = 1\right\} \cong \mathbb{R} \times \mathbb{S}^n,$$

- with the Lorentz metric: $\widehat{\ell}_{\mathcal{E}} = -d\tau^2 + \imath^*\Big(\sum_{j=1}^{n+1} (dx^j)^2 \Big)$, where $\imath : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ denotes the inclusion map,
- the orientation induced by the volume form

$$\Omega_{\widehat{\mathcal{E}}}|_{(\tau,x)} = (d\tau \wedge \imath_x^*(dx^1 \wedge \cdots \wedge dx^{n+1}))|_{T_{(\tau,x)}(\mathcal{E}^{1,n})},$$

- the time-orientation given by requiring that the unit timelike vector field ∂_τ is future-directed.

Outline

Compact Quotients of $\mathcal{E}^{1,n}$: Integral Forms of the 1st Kind

- For $\tau_* > 0$, the subgroup $\mathcal{T}_{\tau^*}$ of $\mathbb{R} \times \mathrm{SO}(n+1)$ generated by $(\tau, \mathrm{y}) \mapsto (\tau + \tau_*, \mathrm{y})$ acts properly discontinuously on $\mathcal{E}^{1,n}$.
- The action preserves the Lorentz structure: the Lorentz metric $\widehat{\ell}_{\mathcal{E}}$, the volume element $\Omega_{\widehat{\mathcal{E}}}$, and time-orientation.
- The quotient $\mathcal{E}^{1,n}/\mathcal{T}_{\tau^*}$ has a unique Lorentz structure, so that the covering map $\pi_{\tau_*} : \mathcal{E}^{1,n} \to \mathcal{E}^{1,n}/\mathcal{T}_{\tau^*}$ is a local isometry preserving orientation and time-orientation.

Definition (Integral Compact Form of the 1st Kind)

- If $\tau_* = 2k\pi$, k a positive integer, we call $\mathcal{E}^{1,n}_{\mathrm{I},k} := \mathcal{E}^{1,n}/\mathcal{T}_{\tau_*}$ the integral compact form of the 1st kind and index k.
- For $k = 1$, we obtain the standard compact form $\mathcal{E}^{1,n}_{\mathrm{I}}$.
- The integral forms $\mathcal{E}^{1,n}_{\mathrm{I},k}$ are all diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^n$.

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Compact Quotients of $\mathcal{E}^{1,n}$: Integral Forms of the 2nd Kind

- Suppose n is odd. The map $\mathrm{T}'_{\tau_*} : (\tau, \mathrm{y}) \mapsto (\tau + \tau_*, -\mathrm{y})$ is an isometry of $\mathcal{E}^{1,n}$ preserving orientation and time-orientation.
- The subgroup $\mathcal{T}'_{\tau^*}$ generated by T'_{τ_*} acts on $\mathcal{E}^{1,n}$ properly discontinuously and preserves the Lorentz structure of $\mathcal{E}^{1,n}$.
- The quotient $\mathcal{E}^{1,n}/\mathcal{T}'_{\tau^*}$ has a unique Lorentz structure, so that the covering $\pi'_{\tau_*} : \mathcal{E}^{1,n} \to \mathcal{E}^{1,n}/\mathcal{T}'_{\tau^*}$ is a local isometry preserving orientation and time-orientation.

Definition (Integral Compact Form of the 2nd Kind)

- If $\tau_* = (2k+1)\pi$, $k \geq 0$ an integer, we call $\mathcal{E}^{1,n}_{\mathrm{II},k} := \mathcal{E}^{1,n}/\mathcal{T}'_{\tau_*}$, the integral compact form of the 2nd kind with index k.
- For $k = 0$, we obtain the standard compact form $\mathcal{E}^{1,n}_{\mathrm{II}}$.
- The integral forms $\mathcal{E}^{1,n}_{\mathrm{II},k}$ are diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^n$.

Compact Quotients of $\mathcal{E}^{1,n}$: Integral Forms of the 2nd Kind

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Outline

Theorem A: The Canonical Covering of $\mathrm{O}_+^{\uparrow}(2, n+1)$

- $\widehat{\mathrm{O}}_+^{\uparrow}(2, n+1) := \left\{(\mathbf{X}, \tau) \in \mathrm{O}_+^{\uparrow}(2, n+1) \times \mathbb{R} \mid \psi(\mathbf{X}) = e^{i\tau}\right\}$ *is an embedded hypersurface of* $\mathrm{O}_+^{\uparrow}(2, n+1) \times \mathbb{R}$.
- $\widehat{\mathrm{O}}_+^{\uparrow}(2, n+1)$ *is diffeomorphic to* $\Omega_{\mathrm{IV}} \times \mathbb{R} \times \mathrm{SO}(n+1)$.
- *The multiplication* $(\mathbf{X}, \tau) \star (\mathbf{X}', \tau') = (\mathbf{X}\,\mathbf{X}', \tau + \tau' + \zeta(\mathbf{X}, \mathbf{X}'))$ *gives* $\widehat{\mathrm{O}}_+^{\uparrow}(2, n+1)$ *a Lie group structure with*
  - *neutral element* $(I_{n+3}, 0)$,
  - *inverse* $(\mathbf{X}, \tau)^{-1} = (\mathbf{X}^{-1}, -\tau - \zeta(\mathbf{X}, \mathbf{X}^{-1}))$.
- *The map* $\sigma : \widehat{\mathrm{O}}_+^{\uparrow}(2, n+1) \to \mathrm{O}_+^{\uparrow}(2, n+1)$, $\sigma(\mathbf{X}, \tau) = \mathbf{X}$, *is a covering homomorphism of Lie groups.*
- *The center* $\widehat{\mathrm{Z}}(2, n+1)$ *of* $\widehat{\mathrm{O}}_+^{\uparrow}(2, n+1)$ *is*
  1. $\widehat{\mathrm{Z}}(2, n+1) = \{(I, 2\pi k) \mid k \in \mathbb{Z}\} \cong \mathbb{Z}$, *if n is even*;
  2. $\widehat{\mathrm{Z}}(2, n+1) = \left\{((-1)^k I, \pi k) \mid k \in \mathbb{Z}\right\} \cong \mathbb{Z}_2 \times \mathbb{Z}$, *if n is odd.*

The Classical Bounded Domain of Type IV

- Let
$$\Omega_{\mathrm{IV}} := \left\{ \beta \in \mathbb{R}(n+1,2) \mid I_2 - {}^t\beta\beta > 0 \right\}.$$
- Ω_{IV} is diffeomorphic to
$$\left\{ z \in \mathbb{C}^{n+1} \mid 2\,{}^t z\bar{z} < 1 + |{}^t zz|^2 < 2 \right\}.$$
- Ω_{IV} is star-shaped w.r.t. the origin $\mathrm{O}_{\mathrm{IV}} = 0_{(n+1)\times 2}$. In particular, Ω_{IV} is contractible.
- Ω_{IV} can be identified with the the Grassmannian $\mathrm{G}_2^-(\mathbb{R}^{2,n+1})$ of 2-dimensional negative definite subspaces $\mathbb{V}$ of $\mathbb{R}^{2,n+1}$.

Transitive Action of $\mathrm{O}^{\uparrow}_{+}(2, n+1)$ on the Domain Ω_{IV}

- Write $\mathbf{X} \in \mathrm{O}^{\uparrow}_{+}(2, n+1)$ as $\mathbf{X} = \begin{pmatrix} \mathfrak{a}(\mathbf{X}) & \mathfrak{b}(\mathbf{X}) \\ \mathfrak{c}(\mathbf{X}) & \mathfrak{d}(\mathbf{X}) \end{pmatrix}$, $\mathfrak{a}(\mathbf{X}) \in \mathbb{R}(2, 2)$.
- $\mathrm{O}^{\uparrow}_{+}(2, n+1)$ acts transitively on Ω_{IV} by

$$L_{\mathbf{X}}(\beta) := (\mathfrak{d}(\mathbf{X})\beta + \mathfrak{c}(\mathbf{X}))\,(\mathfrak{b}(\mathbf{X})\beta + \mathfrak{a}(\mathbf{X}))^{-1}.$$

- The isotropy group at the origin $\mathrm{O}_{\mathrm{IV}} := 0_{(n+1)\times 2} \in \Omega_{\mathrm{IV}}$ is

$$\mathrm{SO}(2)\times\mathrm{SO}(n+1) \cong \left\{ \mathbf{S}(\mathrm{r}, \mathrm{R}) = \begin{pmatrix} \mathrm{r} & 0 \\ 0 & \mathrm{R} \end{pmatrix} \;\middle|\; \begin{array}{l} \mathrm{r} \in \mathrm{SO}(2), \\ \mathrm{R} \in \mathrm{SO}(n+1) \end{array} \right\}.$$

- This gives a coset expression for Ω_{IV},

$$\Omega_{\mathrm{IV}} = \mathrm{O}^{\uparrow}_{+}(2, n+1)/\mathrm{SO}(2) \times \mathrm{SO}(n+1).$$

The Canonical Covering of $\mathrm{O}^{\uparrow}_{+}(2, n+1)$

- The canonical projection of $\mathrm{O}^{\uparrow}_{+}(2, n+1)$ onto Ω_{IV},

$$\pi_2 : \mathrm{O}^{\uparrow}_{+}(2, n+1) \to \Omega_{\mathrm{IV}}, \; \mathbf{X} \mapsto L_{\mathbf{X}}(\mathrm{O}_{\mathrm{IV}}) = \mathfrak{c}(\mathbf{X})\mathfrak{a}(\mathbf{X})^{-1},$$

  is a principal bundle over Ω_{IV} with group $\mathrm{SO}(2) \times \mathrm{SO}(n+1)$.
- There is a global cross section of $\pi_2 : \mathrm{O}^{\uparrow}_{+}(2, n+1) \to \Omega_{\mathrm{IV}}$,

$$\mathrm{P} : \Omega_{\mathrm{IV}} \ni \beta \mapsto \mathrm{P}(\beta) =: \begin{pmatrix} \widehat{\mathfrak{a}}(\beta) & \widehat{\mathfrak{b}}(\beta) \\ \widehat{\mathfrak{c}}(\beta) & \widehat{\mathfrak{d}}(\beta) \end{pmatrix} \in \mathrm{O}^{\uparrow}_{+}(2, n+1).$$

- Any $\mathbf{X} \in \mathrm{O}^{\uparrow}_{+}(2, n+1)$ can uniquely be written as

$$\mathbf{X} = \mathrm{P}\left(\pi_2(\mathbf{X})\right) \begin{pmatrix} \psi(\mathbf{X}) & 0 \\ 0 & \Psi(\mathbf{X}) \end{pmatrix}$$

  for $\psi : \mathrm{O}^{\uparrow}_{+}(2, n+1) \to \mathrm{SO}(2)$, $\Psi : \mathrm{O}^{\uparrow}_{+}(2, n+1) \to \mathrm{SO}(n+1)$.

The Canonical Covering of $O_+^{\uparrow}(2, n+1)$

- For each $t \in \mathbb{R}$, let $e^{it} := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in \mathrm{SO}(2)$
- The map $\widehat{\mathfrak{a}} : \Omega_{\mathrm{IV}} \to \mathrm{GL}_+(2, \mathbb{R})$ has the invariance properties:

$$\widehat{\mathfrak{a}}(\mathrm{R}\beta) = \widehat{\mathfrak{a}}(\beta), \quad \widehat{\mathfrak{a}}(\beta \mathrm{r}^{-1})^{-1}\, \mathrm{r}\, \widehat{\mathfrak{a}}(\beta) \in \mathrm{SO}(2),$$

  for each $\beta \in \Omega_{\mathrm{IV}}$, $\mathrm{R} \in \mathrm{SO}(n+1)$, and $\mathrm{r} \in \mathrm{SO}(2)$.
- There exists a unique $\eta : \Omega_{\mathrm{IV}} \times \mathrm{SO}(2) \to \mathbb{R}$, such that

$$\widehat{\mathfrak{a}}(\beta\, \mathrm{r}^{-1})^{-1}\, \mathrm{r}\, \widehat{\mathfrak{a}}(\beta)\, \mathrm{r}^{-1} = e^{i(\eta(\beta, \mathrm{r}))}, \quad \eta(\mathrm{O}_{\mathrm{IV}}, I_2) = 0,$$

  for each $\beta \in \Omega_{\mathrm{IV}}$ and $\mathrm{r} \in \mathrm{SO}(2)$.

The Canonical Covering of $\mathrm{O}^{\uparrow}_{+}(2, n+1)$

- Let $\mathfrak{m} : \Omega_{\mathrm{IV}} \times \Omega_{\mathrm{IV}} \to \Omega_{\mathrm{IV}}$, $\quad \mathfrak{r} : \Omega_{\mathrm{IV}} \times \Omega_{\mathrm{IV}} \to \mathrm{SO}(2)$, and $\mathfrak{R} : \Omega_{\mathrm{IV}} \times \Omega_{\mathrm{IV}} \to \mathrm{SO}(n+1)$ be the smooth maps defined by

$$\mathrm{P}(\beta)\mathrm{P}(\beta') = \mathrm{P}(\mathfrak{m}(\beta, \beta')) \begin{pmatrix} \mathfrak{r}(\beta, \beta') & 0 \\ 0 & \mathfrak{R}(\beta, \beta') \end{pmatrix}, \quad \forall\, \beta,\ \beta' \in \Omega_{\mathrm{IV}}.$$

- Since Ω_{IV} is simply connected, $\exists!$ $\Theta : \Omega_{\mathrm{IV}} \times \Omega_{\mathrm{IV}} \to \mathbb{R}$, such that, $\mathfrak{r}(\beta, \beta') = e^{i[\Theta(\beta,\beta')]}$ and $\Theta(\mathrm{O}_{\mathrm{IV}}, \mathrm{O}_{\mathrm{IV}}) = 0$.
- Let $\zeta : \mathrm{O}^{\uparrow}_{+}(2, n+1) \times \mathrm{O}^{\uparrow}_{+}(2, n+1) \to \mathbb{R}$ be defined by

$$\zeta(\mathbf{X}, \mathbf{X}') = \Theta\left(\pi_2(\mathbf{X}), \Psi(\mathbf{X})\, \pi_2(\mathbf{X}')\, \psi(\mathbf{X})^{-1}\right) + \eta\left(\pi_2(\mathbf{X}'), \psi(\mathbf{X})\right),$$

for each $\mathbf{X}, \mathbf{X}' \in \mathrm{O}^{\uparrow}_{+}(2, n+1)$.

Outline

The Restricted Conformal Group of $\mathcal{E}^{1,n}$, $\mathcal{E}^{1,n}_{\mathrm{I},h}$ and $\mathcal{E}^{1,n}_{\mathrm{II},h}$

Consider the discrete subgroups of the center $\widehat{\mathrm{Z}}(2, n+1)$:

1. $\mathcal{T}_h := \{(I_{n+3}, 2\pi hk) \mid k \in \mathbb{Z}\}$, any $h \geq 1$,
2. $\mathcal{T}'_h := \left\{\left((-1)^k I_{n+3}, \pi(2h+1)k\right) \mid k \in \mathbb{Z}\right\}$, any $h \geq 0$, n odd.

and the corresponding quotient Lie groups

1. $\widehat{\mathrm{O}}^{\uparrow}_{+}(2, n+1)/\mathcal{T}_h$
2. $\widehat{\mathrm{O}}^{\uparrow}_{+}(2, n+1)/\mathcal{T}'_h$.

Theorem B

1. $\mathcal{C}^{\uparrow}_{+}(\mathcal{E}^{1,n})$ *is isomorphic to* $\widehat{\mathrm{O}}^{\uparrow}_{+}(2, n+1)$.
2. $\mathcal{C}^{\uparrow}_{+}(\mathcal{E}^{1,n}_{\mathrm{I},h})$ *is isomorphic to* $\widehat{\mathrm{O}}^{\uparrow}_{+}(2, n+1)/\mathcal{T}_h$.
3. $\mathcal{C}^{\uparrow}_{+}(\mathcal{E}^{1,n}_{\mathrm{II},h})$ *is isomorphic to* $\widehat{\mathrm{O}}^{\uparrow}_{+}(2, n+1)/\mathcal{T}'_h$.

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Idea of the Proof

- $\mathrm{O}_{+}^{\uparrow}(2, n+1)$ acts effectively on $\mathcal{E}_{\mathrm{I}}^{1,n}$ by RCTs.
- $\widehat{\mathrm{O}}_{+}^{\uparrow}(2, n+1)$ acts effectively on $\mathcal{E}^{1,n}$ by RCTs.
- $\mathcal{T}_h$ is the group of deck transformations of $\pi_{I,h} : \mathcal{E}^{1,n} \to \mathcal{E}_{\mathrm{I},h}^{1,n}$.
- $\mathcal{T}_h'$ is the group of deck transformations of $\pi_{II,h} : \mathcal{E}^{1,n} \to \mathcal{E}_{\mathrm{II},h}^{1,n}$.
- The action of $\widehat{\mathrm{O}}_{+}^{\uparrow}(2, n+1)$ descends to effective actions of $\widehat{\mathrm{O}}_{+}^{\uparrow}(2, n+1)/\mathcal{T}_h$ on $\mathcal{E}_{\mathrm{I},h}^{1,n}$, and of $\widehat{\mathrm{O}}_{+}^{\uparrow}(2, n+1)/\mathcal{T}_h'$ on $\mathcal{E}_{\mathrm{II},h}^{1,n}$ by RCTs.
- Use the theorem on the group of RCTs of a Lorentz manifold.

Remark

- $\mathcal{E}^{1,n}_{\mathrm{I},k}$ and $\mathcal{E}^{1,n}_{\mathrm{I},h}$, $k \neq h$, cannot be conformally diffeomorphic.
- $\mathcal{E}^{1,n}_{\mathrm{II},k}$ and $\mathcal{E}^{1,n}_{\mathrm{II},h}$, $k \neq h$, cannot be conformally diffeomorphic.
- $\mathcal{E}^{1,n}_{\mathrm{II},k}$ and $\mathcal{E}^{1,n}_{\mathrm{I},h}$ cannot be conformally diffeomorphic.

Outline

Theorem C

- *Let* $\mathbb{M}$ *be a* $(n+1)$*-dimensional,* $n \geq 2$*, connected, oriented, time-oriented conformal Lorentz manifold such that*

$$\dim(\mathcal{C}_+^{\uparrow}(\mathbb{M})) = \frac{1}{2}(n+3)(n+2).$$

- $\mathbb{M}$ *simply connected* $\Longrightarrow$ $\mathbb{M}$ *is conformally equivalent to* $\mathcal{E}^{1,n}$*.*
- $\mathbb{M}$ *not simply connected, n even,* $\Longrightarrow$ $\mathbb{M}$ *is conformally equivalent to an integral compact form of the 1st kind,* $\mathcal{E}^{1,n}_{\mathrm{I},k}$*.*
- $\mathbb{M}$ *not simply connected, n odd,* $\Longrightarrow$ $\mathbb{M}$ *is conformally equivalent to either*
  1. *an integral compact form of the 1st kind,* $\mathcal{E}^{1,n}_{\mathrm{I},k}$*, or*
  2. *an integral compact form of the 2nd kind,* $\mathcal{E}^{1,n}_{\mathrm{II},k}$*.*