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Codimension two spacelike submanifods through a null hypersurface of a pf-wave

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joint work in progress with Alfonso Romero

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Definition (Brinkmann spacetime)

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- For every $p \in \bar{M}$ we can define the its future as the connected component of the null cone with vertex at p so that $K(p)$ is contained in the clousure of such component.

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- We can consider local coordinates $(u, v, x_1, \ldots, x_n) = (u, v, \mathbf{x})$.
- The metric $\bar{g}$ can be locally written by

$$\mathcal{F}(u,\mathbf{x})du\otimes du+2du\otimes dv+\sum_{i,j}\bar{g}_{i,j}(u,\mathbf{x})dx_i dx_j,$$

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  where $\mathcal{F}$ is a smooth function with no required sign.
- With this coordinates K coincides with the coordinate vector field $\partial_v = \partial/\partial v$.

Definition (plane fronted wave)

Is a Brinkmann spacetime with the form $\bar{M}^{n+2} = \mathbb{R}^2 \times M^n$ and metric

$$\langle , \rangle = \mathcal{F}(u, \mathbf{x}) du \otimes du + 2du \otimes dv + g_M$$

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$\bar{M} = \mathbb{R}^2 \times M$, with coordinates $(u, v, \mathbf{x})$ and metric

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If $M = \mathbb{R}^n$ is the Euclidean space, these spacetimes are called exact pf-waves (pp-waves).

Following

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we obtain:

Levi-Civita connection of $\bar{M}$

On $\bar{M}$, if $V, W \in \mathcal{L}(M)$ we have

i) $\overline{\nabla}_{\partial_u}\partial_u = \frac{1}{2}\overline{\nabla}\mathcal{F} - \tilde{\nabla}\mathcal{F}_u$,

ii) $\overline{\nabla}_V\partial_u = \overline{\nabla}_{\partial_u}V = \frac{1}{2}\, g_M(\tilde{\nabla}\mathcal{F}_u, V)\partial_v$,

iii) $\overline{\nabla}_V W = \tilde{\nabla}_V W$,

iv) $\overline{\nabla}_{\partial_v}\partial_v = \overline{\nabla}_{\partial_v}\partial_u = \overline{\nabla}_{\partial_u}\partial_v = \overline{\nabla}_V\partial_v = \overline{\nabla}_{\partial_v}V = 0$,

where $\tilde{\nabla}\mathcal{F}_u$ denote the grandient on M of the function $\mathcal{F}_u(\mathbf{x}) := \mathcal{F}(u, \mathbf{x})$ for every $\mathbf{x} \in M$, $\overline{\nabla}\mathcal{F}$ denote the gradient of the function $\mathcal{F}$ on $\bar{M}$, and $\tilde{\nabla}$ is the Levi-Civita connection of M.

Riemann curvature tensor of $\bar{M}$

The only non zero components of $\overline{\mathrm{R}}$ are

i) $\overline{\mathrm{R}}(V, \partial_u)\partial_u = -\frac{1}{2}\tilde{\nabla}_X \tilde{\nabla}\mathcal{F}_u$ and

ii) $\overline{\mathrm{R}}(V, \partial_u)W = \frac{1}{2}\tilde{\mathrm{Hess}}(\mathcal{F}_u)(V, W)\partial_v$,

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Ricci tensor

The only non zero components of $\overline{\mathrm{Ric}}$ are

$$\overline{\mathrm{Ric}}(V, W) = \mathrm{Ric}_M(V, W) \quad \text{and} \quad \overline{\mathrm{Ric}}(\partial_u, \partial_u) = -\frac{1}{2}\tilde{\Delta}\mathcal{F}_u,$$

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A pf-wave $\bar{M}$ satisfies the timelike convergence condition (TCC) if, and only if,

$$\tilde{\Delta}\mathcal{F}_u \leq 0 \quad \text{and} \quad \mathrm{Ric}_M \geq 0.$$

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$$X = X^\top + X^\perp,$$

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- We denote A_ζ the shape operator associated to the normal vector field ζ,
$$\langle A_\zeta X, Y\rangle = \langle \mathrm{II}(X, Y), \zeta\rangle$$
with II the second fundamental form of Σ.

- As usual, we define the mean curvature vector field by

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Definition

We say that Σ is

i) trapped if $\mathbf{H}$ is timelike, $\langle \mathbf{H}, \mathbf{H} \rangle < 0$,

ii) marginally trapped if $\mathbf{H}$ is null, $\langle \mathbf{H}, \mathbf{H} \rangle = 0$, $\mathbf{H} \neq 0$ and

iii) weakly trapped if $\mathbf{H}$ is causal, $\langle \mathbf{H}, \mathbf{H} \rangle \leq 0$, $\mathbf{H} \neq 0$.

Null hypersurface

If we fix $u = u_0$, $u_0 \in \mathbb{R}$, then $\{u_0\} \times \mathbb{R} \times M$ is a null hypersurface of $\bar{M}$ that we denote by $\mathbb{R}_{u_0} \times M^n$.

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Observe that $g = u_0$ constant if and only if Σ factorizes through $\mathbb{R}_{u_0} \times M^n$.

- We compute

$$\nabla g = \partial_v^\top \quad \text{and} \quad \Delta g = \mathrm{tr}(A_{\partial_v^\perp}) = n\langle \mathbf{H}, \partial_v \rangle.$$

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If Σ factorizes through a null hypersurface $\mathbb{R}_{u_0} \times M^n$, then the projection of Σ on M is a local isometry.

Corollary

Let Σ factorize through $\mathbb{R}_{u_0} \times M^n$. If Σ is complete and non-compact, then the projection $\phi : \Sigma \to M$ is a Riemannian covering map. In addition, if M is simply connected, then ϕ is a global isometry.

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Let $\psi : \Sigma \to \bar{M}$ a codimension two spacelike submanifold which factorizes through the null hypersurface $\mathbb{R}_{u_0} \times M^n$. Then,

$$\xi = \partial_v \quad \text{and} \quad \eta = -\partial_u^{\perp} + \frac{1}{2}\mathcal{F}\partial_v$$

are two globally defined normal vector fields that satisfy

$$\langle \xi, \xi \rangle = 0, \quad \langle \eta, \eta \rangle = - \mid \partial_u^{\top} \mid^2 \leq 0, \quad \text{and} \quad \langle \xi, \eta \rangle = -1.$$

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That is, $\{\xi, \eta\}$ is a globally defined normal frame on Σ with ξ null, η timelike and $\langle \xi, \eta \rangle = -1$.

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Proposition

Let $\psi : \Sigma \to \mathbb{R}_{u_0} \times M^n \subset \bar{M}$ a codimension two spacelike submanifold wich factorizes through a null hypersurface $\mathbb{R}_{u_0} \times M^n$. Then Σ is marginally trapped, whenever $\mathbf{H}$ does not vanish on Σ.

Slice

If we fix $u = u_0$ and $v = v_0$, $u_0, v_0 \in \mathbb{R}$, then $\{u_0, v_0\} \times M$ is a slice of $\bar{M}$ that we denote by $(u_0, v_0) \times M$.

When $\psi(\Sigma) \subset (u_0, v_0) \times M$ we say that Σ is also a slice.

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Lemma

The family of slices is a distinguished class of totally geodesic codimension two spacelike submanifolds of $\bar{M}$

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- By the compactness of Σ if follows $\Delta g = 0$ and g is constant. That is, Σ factorizes through a null hypersurface.
- Then $\mathbf{H} = \frac{1}{n}\Delta h \partial_v$ and, since $\mathbf{H}$ never vanishes,

$$\Delta h \leq 0 \quad \text{or} \quad \Delta h \geq 0,$$

but it can not happen.

Theorem

Let $\psi : \Sigma \to \bar{M}$ be a codimension two spacelike submanifold in a pf-wave with signed $\langle \mathbf{H}, \partial_v \rangle$ (in particular if $\mathbf{H} = 0$). Then, if Σ is compact,the following assertions are satisfied:

- i) *Σ factorizes through a null hypersurface $\mathbb{R}_{u_0} \times M^n$.*
- ii) *Σ is isometric to M, and therefore, M is compact.*
- iii) *Σ is a slice, and hence, it is totally geodesic.*

Theorem

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Theorem

Let $\psi : \Sigma \to \bar{M}$ be a complete codimension two spacelike submanifold with zero mean curvature which factorizes through a null hypersurface $\mathbb{R}_{u_0} \times M^n$. If $\bar{M}$ satisfies the TCC and the function h is bounded from above or from below, then Σ is a slice.

Index

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- Maybe focus on the compact case.

Thank you for your attention

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