# Solitons on Wormholes

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work with Piotr Bizoń, Gary Gibbons, Michal Kahl, Michal Kowalczyk

# Sine–Gordon kink

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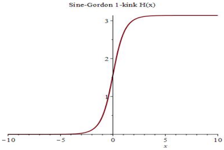

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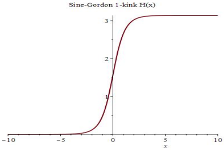


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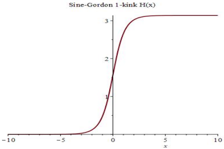


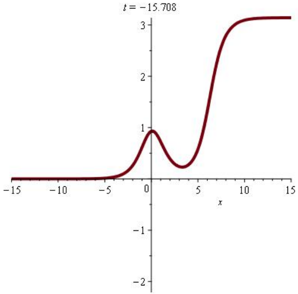

- Topologically stable: $N \equiv \pi^{-1}(H(\infty) - H(-\infty)) = 1$.
- Linear stability

$$\begin{aligned} \phi(t,x) &= H(x) + \epsilon u(x,t), \quad u(x,t) = e^{i\omega t}\psi(x) \\ L(\psi) &= \omega^2\psi, \quad L = -{\partial_x}^2 - 4\text{sech}^2(\sqrt{2}x) + 2. \end{aligned}$$

Spec$(L) = \{0\} \cup [2, \infty)$.

# Wobbling kink

SG kink is not asymptotically stable: time dependent wobbling kinks (time–dependent solitons resulting from integrability of Sine–Gordon)



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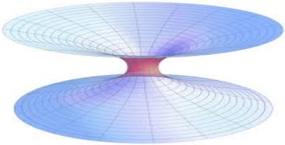
- Prove existence of globally regular static kinks for any kink number $N$.
- Linear stablity: $\exists$ internal oscillation modes.
- A step towards the soliton resolution conjecture: Solutions of dispersive wave equations asymptotically resolve into a superposition of localised structures (solitons, black holes, ...) and radiation. (e.g. Terence Tao's blog *Structure and randomness in PDE*).

# Ellis–Bronnikov wormhole

- $(t, r) \in \mathbb{R} \times \mathbb{R}, (\theta, \phi) \in S^2$. Wormhole with area $4\pi a^2$

$$g = -dt^2 + dr^2 + (r^2 + a^2)(d\theta^2 + \sin^2\theta d\phi^2)$$



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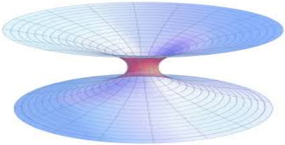

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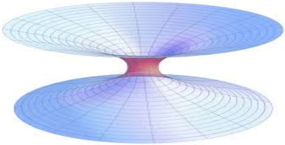

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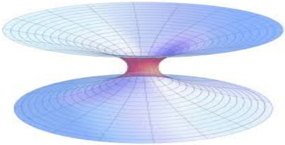

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- Sine–Gordon $\Box_g(\phi) = \sin(2\phi)$. Finite energy solutions s.t

$$\phi \sim N_{\pm}\pi \text{ as } r \to \pm\infty, \quad N \equiv \pi^{-1}(N_+ - N_-) \neq 0.$$

# Radial Sine–Gordon on a wormhole

$$\phi_{rr} - \phi_{tt} + \frac{2r}{r^2 + a^2}\phi_r = \sin(2\phi)$$

- Not integrable (e.g. fails the Painleve test), not scale invariant if $a \neq 0$, dispersive (a shadow of higher dimensions).

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$$E[\phi] = \int_{\mathbb{R}} \Big(\frac{1}{2}{\phi_t}^2 + \frac{1}{2}{\phi_r}^2 + \sin^2\phi\Big)(r^2 + a^2)dr.$$

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- Finite energy solutions: $\phi(t, -\infty) = 0, \phi(t, \infty) = N\pi$.

# Static kinks

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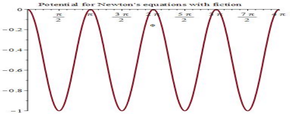
- Proof: $(*)$ =the Newton equation (with time $r$) of a particle in a reversed potential $-\sin^2 H$ with time–dependent friction.



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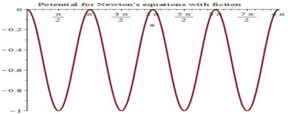
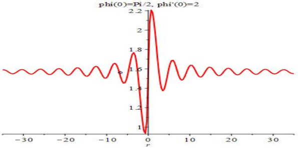



- Generic initial velocity $\phi'(0)$ with $\phi(0) = N\pi/2$.



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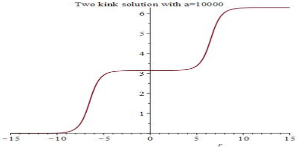
$$\begin{aligned} H_N(r) &\sim H_{SG}(r-(2k-1)R) + \cdots + H_{SG}(r-R) + H_{SG}(r+R) \\ &+ \cdots + H_{SG}(r+(2k-1)R) \end{aligned}$$

where $R = \sqrt{2}/4W(32\coth(3\sqrt{2})a^2/N^2)$, and $W(x)e^{W(x)} = x$.



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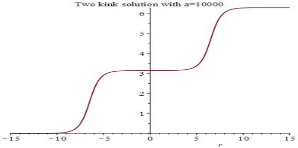


- Small $a$

$$H_N(r) \sim N\left(\frac{\pi}{2} + \arctan(r/a)\right).$$

- Let $\phi(r,t) = H_N(r) + \epsilon(r^2+a^2)^{-1/2}u(r,t)$. Then $u_{tt} + L_N(u) = 0$

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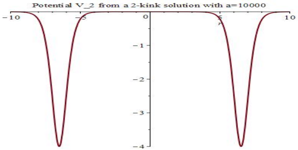

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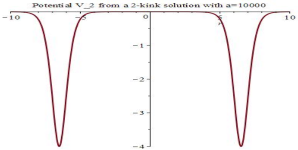


- As $a$ decreases the eigenvalues disappear one by one into the continuous spectrum. For $N=1$ the continuous spectrum for $a < a_c \sim 0.53$.

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