CONVEXIFYING POSITIVE POLYNOMIALS AND SUMS OF SQUARES APPROXIMATION

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Abstract. We show that if a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ is nonnegative on a closed basic semialgebraic set $X = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_r(x) \geq 0\}$, where $g_1, \dots, g_r \in \mathbb{R}[x_1, \dots, x_n]$, then f can be approximated uniformly on compact sets by polynomials of the form $\sigma_0 + \varphi(g_1)g_1 + \cdots + \varphi(g_r)g_r$, where $\sigma_0 \in \mathbb{R}[x_1, \dots, x_n]$ and $\varphi \in \mathbb{R}[t]$ are sums of squares of polynomials. In particular, if X is compact, and $h(x) := R^2 - |x|^2$ is positive on X, then $f = \sigma_0 + \sigma_1 h + \varphi(g_1)g_1 + \cdots + \varphi(g_r)g_r$ for some sums of squares $\sigma_0, \sigma_1 \in \mathbb{R}[x_1, \dots, x_n]$ and $\varphi \in \mathbb{R}[t]$, where $|x|^2 = x_1^2 + \cdots + x_n^2$. We apply a quantitative version of those results to semidefinite optimization methods.

Let X be a convex closed semialgebraic subset of $\mathbb{R}^n$ and let f be a polynomial which is positive on X. We give necessary and sufficient conditions for the existence of an exponent $N \in \mathbb{N}$ such that $(1 + |x|^2)^N f(x)$ is a convex function on X. We apply this result to searching for lower critical points of polynomials on convex compact semialgebraic sets.

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