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Report on the habilitation achievement "Deformations, degenerations, and homotopy types of algebraic varieties" by Dr. Piotr Achinger

Dr. Piotr Achinger is a researcher in mathematics who has specialized in algebraic geometry. He got master degrees in mathematics and computer science from the University of Warsaw. He pursued his mathematical career, writing a PhD thesis under the supervision of Prof. Arthur Ogus at the University of Berkeley, California, USA. (Prof. Ogus is a renowned scientist who gave a talk at the International Congress of Mathematics in Warsaw, 1983.) Soon afterwards, he became an assistant professor at the Institute of Mathematics of the Polish Academy of Sciences in Warsaw. He also gained some experience abroad as a postdoctoral researcher at the Institut des Hautes Études Scientifiques in Bures-Sur-Yvette, France, as a visitor to the Max-Planck-Institute in Bonn, Germany, and as a program participant at the MSRI, Berkeley, USA.

Dr. Achinger selected five papers for the habilitation achievement. His text, entitled "Deformations, degenerations, and homotopy types of algebraic varieties", provides a decent introduction to the research topics of the papers and detailed, instructive, and well-written summaries of these papers. All of the papers appeared in well-known and recognized international journals. In fact, "Journal für die reine und angewandte Mathematik", "Geometry & Topology", and "Journal of the European Mathematical Society" are very good to excellent journals, and "Inventiones Mathematicae" is one of the top journals in pure mathematics. Four of the five papers were written with co-authors. (It is a pity that Dr. Achinger has not commented on his input to those papers. This makes it a bit difficult to assess his merits.)

In the study of algebraic varieties over the complex numbers, one often uses topological or even analytic methods, exploiting the fact that a complex algebraic variety determines in a natural way a topological space and a complex analytic space. Making use of embeddings into the field of complex numbers, it is possible to extend these tools to varieties defined over a field of characteristic zero. For varieties which are defined over a field of positive characteristic, such a direct approach is not available. In the introduction to the habilitation achievement, Dr. Achinger explains several approaches to and techniques for studying varieties in a "topological fashion". The different approaches and techniques are then illustrated by the five papers in the habilitation achievement.

Lifting to characteristic zero. - The most straightforward way to apply characteristic zero techniques to a (proper) variety X which is defined over a field k of positive characteristic is to *lift X to characteristic zero*. For this, one has to find a local domain $(R,\mathfrak{m})$, such that $R/\mathfrak{m}=k$ and the quotient field of R has characteristic zero, as well as a proper and flat morphism $\mathfrak{X}\longrightarrow\mathrm{Spec}(R)$ whose fiber over $\mathfrak{m}$ is isomorphic to X. (One lifts X to the ring R of characteristic zero (in a flat way) via the residue homomorphism $R\longrightarrow k$.) It was observed by Serre that not every proper variety, say over $\mathbb{F}_p$, p a prime number, admits a lifting to characteristic zero.

In [Hab1], Achinger and Zdanowicz show that certain rational smooth projective varieties over finite fields cannot be lifted to characteristic zero. The varieties are constructed in an explicit manner as iterated blow ups of a projective space and previously appeared in the context of the Drinfeld upper half space. Using the well-known technique of blowing down deformations, the authors reduce the property of liftability to a problem in the representation theory of matroids. (It is a bit unfortunate that there are typos in Lemma 3.6 and the subsequent discussion which make it difficult to understand the result on matroids at a first reading.) Due to the explicit nature of the examples, one can derive important information on the cohomology of the examples, e.g., on their virtual motive in the Grothendieck ring of varieties. Together with other results on (non-)liftability of algebraic varieties, these examples are an important ingredient toward a better understanding of the condition of liftability to characteristic zero.

In the case that liftings to characteristic zero do exist, they will in general not be unique and one is faced with the problem of understanding the different possible liftings or of finding a canonical lifting. Serre-Tate theory deals with the deformation theory of ordinary abelian varieties over a perfect field of positive characteristic. It implies that such an abelian variety has a canonical lifting to characteristic zero over the ring of Witt vectors. Nygaard developed a similar theory in the realm of K3 surfaces which was generalized to the case of threedimensional varieties with trivial canonical bundle by Ward. In a part of [Hab2], these findings were generalized to varieties of arbitrary dimension with trivial canonical bundle. The larger part of the paper is concerned with liftings of such varieties modulo p^2. This theory is closely related to the theory of Frobenius splittings, initiated by Mehta and Ramanathan. The authors made interesting observations relating Frobenius splittings to other objects. Since Calabi-Yau varieties play an important role in mathematical physics and the classification theory of algebraic varieties and often have interesting arithmetic properties, the extension of Serre-Tate theory to this class of varieties is of great interest.

Article [Hab3] also studies liftings of algebraic varieties modulo p^2 and addresses the question which varieties admit a lifting modulo p^2 together with a lifting of the Frobenius morphism. Varieties admitting such liftings are called *F-liftable*. The authors of [Hab3] prove several remarkable results about (smooth projective) *F*-liftable varieties. These include the Bott vanishing theorem and a structure result for homogeneous *F*-liftable varieties. The results encouraged the authors to formulate a conjecture giving a hypothetical structure result for *F*-liftable varieties. They supply some evidence for their conjecture and prove that it implies a conjecture by Occhetta and Wiśniewski. The latter conjecture deals with smooth projective varieties X in characteristic zero for which there exist a complete toric variety and a surjection of that toric variety onto X. There are also results for pairs. (The theory of pairs is relevant, e.g., for the classification theory of affine varieties.) The results of the paper illustrate in a convincing way the relevance of the theory of *F*-liftable varieties. The relation to the conjecture by Occhetta and Wiśniewski shows that questions about liftability also allow to reduce questions in characteristic zero to results in positive characteristic.

Homotopy theory. - Another interesting approach is to adapt techniques from homotopy theory to algebraic geometry. For example, Grothendieck introduced the étale fundamental group based on the theory of étale covers of algebraic varieties. More generally, Artin and Mazur proposed an étale homotopy theory for schemes. Finally, $\mathbb{A}^1$-homotopy theory unites scheme theory and homotopy theory. An interesting and crucial fact is that the étale fundamental group of the affine line over a field of positive characteristic is far from being trivial. (This is unlike the situation in $\mathbb{A}^1$-homotopy theory where the affine line is made contractible.)

Paper [Hab4] is devoted to $K(\pi,1)$-schemes. It is in my opinion the strongest paper in the collection that constitutes the habilitation achievement. The main result states that every connected affine scheme over the field $\mathbb{F}_p$, p a prime number, is a $K(\pi,1)$-scheme. This is a very elegant and fundamental result. It shows that étale homotopy theory for connected affine schemes over $\mathbb{F}_p$ reduces to the étale fundamental group and shows how different étale homotopy theory is in characteristic zero and in positive characteristic. Achinger also proves a similar statement for certain noetherian affinoid adic spaces, generalizing a result of Fields medalist Peter Scholze. The proof of the main result is technically very involved and includes a beautiful construction in the inductive step, called *Bertini theorem* by the author.

Paper [Hab5], which Achinger co-authored with Mattia Talpo, deals with homotopy theory for algebraic varieties and rigid analytic spaces over the field, say, $\mathbb{C}((t))$. This is not a field of positive characteristic, but a field with a non-archimedean valuation. In this impressive work, the authors introduce Betti realization functors on the categories of schemes that are locally of finite type over $\mathbb{C}((t))$ and of smooth rigid analytic spaces over $\mathbb{C}((t))$. The target is a certain category of spaces over the circle S^1. The construction of these functors is technically difficult. Yet, Achinger and Talpo manage to discuss a nice geometric example. In fact, they prove that the Betti realization of the non-archimedean Hopf surface of Voskuil is homotopy equivalent to the "classical" Hopf surface. The latter is a fundamental example of a minimal compact complex surface which does not admit a Kähler metric and, therefore, appears in the classification of compact complex surfaces.

A common feature of the articles in the habilitation achievement is that they are all technically brilliant, witness the vast knowledge of Dr. Achinger, feature interesting ideas, and contain results that are of fundamental importance and can be understood and appreciated by any algebraic geometer. The habilitation achievement is apart from several typos well-written and lucid. The review of [Hab4] in MathSciNet confirms that the same is true of the original articles. The broad variety of techniques appearing in the habilitation achievement is very impressive, too.

Besides the articles discussed in the habilitation achievement, Dr. Achinger published several other articles which also appeared in excellent mathematical journals such as "Compositio Mathematica" and "Journal of Algebraic Geometry". It is remarkable that some of these articles were finished before he obtained his PhD. As the articles of the habilitation achievement, these articles contain elegant and significant results. E.g., Achinger's paper in "International Mathematics Research Notices (IMRN)" contains a nice characterization of connected smooth projective varieties over an algebraically closed field of positive characteristic in terms of properties of the Frobenius morphism, based on work of Thomsen. Together with Zdanowicz, Achinger constructed in the appendix to [Zda21] the first examples of varieties with trivial canonical bundle which cannot be lifted to characteristic zero.

Dr. Achinger has been invited to numerous international conferences, e.g., in Japan and the United States. This shows that his work is taken note of and recognized. It is also noteworthy that he has already obtained quite a number of prestigious awards and scholarships, most notably the ERC starting grant "Homotopy Theory of Algebraic Varieties". He also co-organized two editions of the famous GAEL series of conferences for young researchers and several interesting scientific events in Poland. It is also remarkable that Dr. Achinger is engaged in the popularization of mathematics.

The teaching experience of Dr. Achinger is somewhat limited. (This is probably due to the fact that he is working at the academy.) In fact, he has listed only one rather advanced course in his documents. He supervised an undergraduate thesis as well as a master thesis and co-advised three PhD theses. (Again, there is no information on the share of Dr. Achinger in these supervisions.) Together with his work on the popularization of science, this experience is still sufficiently relevant.

With his participation in the IM PAN scientific council and the board of the Warsaw division of the Polish Mathematical Society, Dr. Achinger has also gained basic insight into the administrative work of a researcher.

The scientific record of Dr. Piotr Achinger is excellent. The scope of his research is broad, the applied techniques sophisticated, and the results meaningful. He has an independent research profile, e.g., in his recent research on the topology of algebraic varieties and rigid analytic spaces over non-archimedean fields he opened up a new line of investigation, and he is internationally visible. Dr. Achinger has at least basic experience in all the fields of activity of a professor. Against this background, I strongly suggest to the Scientific Council to approve the habilitation achievement of Dr. Piotr Achinger and to confer him the corresponding titles.

Sincerely yours

A. Schmitt

Alexander Schmitt