Report on the Ph. D. thesis manuscript

“Asymptotic properties of Robinson-Schensted algorithm and Jeu de Taquin”

by Łukasz Maślanka

The results presented in the thesis focus on the study of asymptotic behaviors of some classical combinatorial constructions which are ubiquitous in representation theory of the symmetric groups or the linear Lie algebras. It is in particular well-known, since the works by Schur and Frobenius, that the representation theory of the symmetric group is controlled by the combinatorics of partitions and standard tableaux. In the seventies it was also observed by Lascoux and Schützenberger that a simple insertion procedure on tableaux introduced by Schensted (next generalized by Robinson and Knuth) was a key tool in order to give a simple combinatorial proof of the Littlewood-Richardson rule yielding the tensor product multiplicities associated to the irreducible representations of $\mathfrak{gl}_n$. Later, Lusztig uses the same combinatorial tool to get a natural decomposition of the representations of the Hecke algebras of the symmetric groups into cells. The Jeu de Taquin (or sliding algorithm) can be regarded as an alternative construction of the insertion (or bumping) procedure. The main difference here is the possibility to consider tableaux whose shapes are no longer partition diagrams (Young diagrams).

As initially observed by Schensted, the insertion algorithm on standard tableaux also permits to study easily the longest increasing or decreasing sequences in a given sequence of real numbers. This opened a new and fascinating research domain where these sequences are chosen randomly. Since then, this domain is at the heart of an intense research activity and constitutes the main topic of the present thesis. One of the most classical question in this area is the Ulam problem of the asymptotic behavior of the longest decreasing subword in a word whose letters are chosen uniformly and independently in the interval $[0, 1]$. It is worth noticing here that the popularity of this type of questions is mainly due to their connections with theoretical physics models. For example, the Ulam problem is strongly connected to the Hammersley problem modeling the evolution of particles in the quarter plane.

After a brief introduction where the previous combinatorial construction (RSK algorithm, Jeu de Taquin) are recalled and illustrated by examples together with their connections with representation theory, the thesis is divided in three chapters devoted to the exposition of the new results obtained. These results of Łukasz Maślanka were established in collaboration with his Ph.D. supervisor and also with M. Marciniak for the results presented in Chapters 2 and 3. Before giving a short description of the main achievements of the thesis, let me mention that I thought it was very well written, with sufficiently many examples to understand the various combinatorial definitions or constructions that were introduced. I also appreciated that the limits of the results obtained were clearly indicated with some perspectives for future developments or connected research directions.
The main topic of Chapter 2 is a generalisation (Theorem 2.1.1) to multiple rows of a result by Aldous and Diaconis giving a limit theorem for the growth of the longest row of random tableaux obtained by applying the RSK procedure to the previous random word input (which corresponds to the Plancherel measure on the Young lattice). In Chapter 3, Łukasz Maślanka studies bumping routes for random standard tableaux of size n (with the Plancherel measure) when a new letter (typically $\frac{n}{m}$ where $m \leq n$) is inserted in the bottom row. It is shown in Theorem 3.1.5 that after a projective change of coordinates, these bumping routes converge (when m tends to infinity) in distribution to the Poisson process on $\mathbb{R}_+$ with the unit density. This extends in particular previous results obtained by Romink and Śniady in which more restrictive constraints on the convergence regime were imposed. The last chapter of the thesis is probably the most substantial, not only because it covers the whole second half of the manuscript, but also since it contains very nice and interesting results in connection with TASEP models. It is devoted to a detailed analysis of the limit shapes of evacuation and sliding paths in a uniformly random standard tableau. Given such a tableau, one can indeed remove its southwest corner, apply the Jeu de Taquin procedure and record the path corresponding to the bumping route so obtained. It is proved in the thesis that the probability distribution of these routes converge in probability to particular random meridians connecting the two opposite corners of the tableau (see in particular Theorem 4.2.3). A very interesting aspect of these slidings paths are their connections with a natural TASEP model where two classes of particles are considered. The dictionary between both combinatorial models is made explicit in the thesis. In particular

the boxes of the standard tableaux correspond to the first class of particles whereas the route followed by the empty box encodes the behavior of the second class of particles. It then becomes possible to apply the asymptotic results on bumping routes to study the asymptotic behavior of this second class particles system.

In addition to the previously mentioned theorems, the thesis offers a lot of variations, complementary results or possible generalizations which cannot be all detailed here. One might also mention that the proofs of the results obtained are highly non trivial even when they are based on existing techniques. This is the case in particular in Chapter 2 where delicate manipulations on total variation distances are required. In this Chapter 2, I also found particularly interesting the arguments used and the new light they shed on the Ulam problem or the Hammersely process. The proofs of the results presented in Chapter 3 require also a lot of work. In the course of this chapter various reformulations or satellite results are indeed established. One might evoke the augmented growth process of § 3.3.6 which is reminiscent of the classical Plancherel growth process on the Young lattice. Here ordinary Young diagrams are replaced by augmented ones (pairs $(\lambda, \square)$ where λ is a Young diagram and $\square$ one of its outer corners) yielding an augmented Young lattice. It is then explained in Theorem 3.3.5 how to define a Markov chain on this augmented Young lattice related to the study of the bumping routes considered. Chapter 4 also contains numerous interesting new ideas or material. For example, it is explained how the assumption on the square shaped tableaux on which the evacuation is performed can be relaxed in order to take into account tableaux with L-shape. I also found the notions of *pool* and *single or multiple surfer* scenarios (in addition to their funny aspects) very helpful and relevant to help the reader understand the main ideas used here. This is in particular the case in § 4.6.2 where the random behavior of the previous multisurfer model is connected to the representation theory of the symmetric group, its asymptotic characters and the Jucys-Murphy elements.

To conclude, I would say that from a scientific point of view, the achievements of the thesis of Łukasz Maślanka are absolutely relevant and worthy for publications in mathematical journals of excellent level. I was impressed by the variety of the techniques and mathematical background used in the thesis, but also by the high level of difficulty in the proofs presented. As far as I could check, the arguments are correct and complete. They demonstrate that Ł. Maślanka already masters many aspects of deepful modern mathematical theories and has gained a very good knowledge of the literature in the field . I also appreciated that directions for future works were evoked in each of the main chapters of the thesis. It is undoubtedly a very good thesis and a strong contribution to this rapidly expanding research area at the crossroad between probability, combinatorics and algebra. Taking all this into account, I am pleased to recommend to the PhD committee to support the motion to confer the degree of doctor of mathematical sciences upon Mr Łukasz Maślanka.

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